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The quantum double as quantum mechanics

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We introduce $*$-structures on braided groups and braided matrices. Using this, we shov that the quantum double $D(U_q(\mathrm{su}_2))$ can be viewed as the quantum algebra of observable of a quantum particle moving on a hyperboloid in q-Minkowski space (a three-sphere i the Lorentz metric), and with the role of angular momentum played by $U_q(\mathrm{su}_2)$. Thi provides a new example of a quantum system whose algebra of observables is a Hop algebra. Furthermore, its dual Hopf algebra can also be viewed as a quantum algebra o observables, of another quantum system. This time the position space is a q-deformation o $\mathrm{SL}(2,\mathbb{R})$ and the momentum group is $U_q(\mathrm{su}_2^*)$ where su_2^* is the Drinfeld dual Lie algebra o su_2. Similar results hold for the quantum double and its dual of a general quantum grouį

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1. Introduction

One of the most important quantum group constructions is Drinfeld's quan tum double [1]. It has a rich algebraic structure and, moreover, plays an impor tant (though not fully understood) role in quantum inverse scattering as som kind of quantum dressing transform [2]. It has also been proposed as a kind o 'complexification', for example the double of $U_q(\mathrm{su}_2)$ has been proposed as a quantum Lorentz group [3]. In both of these contexts the quantum double play the physical role of a kind of generalized symmetry.

In this paper we give a new physical interpretation of the quantum double as the algebra of observables of a quantum mechanical system. A further in terpretation as a quantum frame bundle is given in [4]. These interpretation are made possible by results about the algebraic structure of the quantum dou ble in [5] and [6], to which the present paper is a sequel. The main result o

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these works is that the quantum double of a true quantum group (with universal R-matrix or quasitriangular structure [1]) has the structure of a semidirect product. This opens up the possibility of a quantum mechanical interpretation in the context of Mackey quantization [7,8] and its natural generalization to quantum groups [9]. Of course, for a quantum mechanical interpretation, we need to extend the semidirect product result for the quantum double to the level of $*$-algebras, and this is the main goal of the present paper from a mathematical point of view. In doing this, we will have to study $*$-structures on quantum groups in relation to their quasitriangular structure $\mathcal{R}$ as well as $*$-structures on certain associated braided groups [10,11]. This is the topic of section 2. We distinguish two natural possibilities, namely $\mathcal{R}^{*\otimes *} = \mathcal{R}_{21}$ (the real case) and $\mathcal{R}^{*\otimes *} = \mathcal{R}^{-1}$ (the antireal case). The first possibility has been noted some time ago in a classification by Lyubashenko [12] and applies to the compact forms of the standard quantum deformations $U_q(g)$. The second possibility seems to be more novel, and is known to apply for example to $U_q(\mathrm{sl}(2,\mathbb{R}))$. Both are needed in the paper.

The quantum mechanical picture of the double is obtained in section 3. In fact, we have already asked in a series of papers the following question: when is the algebra of observables of a quantum system a Hopf algebra (and if so, what does it mean physically)? In answer to this question we found a large class of homogeneous spaces such that the algebras of observables of quantum mechanics on them (via Mackey quantization) were indeed Hopf algebras. Not any homogeneous space satisfies this and the ones that do so arise in pairs from the factorization of any group into two subgroups (the two factors then act on each other giving two matching homogeneous spaces). Moreover, the dual Hopf algebra of the quantization of one homogeneous space is the Hopf algebra of observables of quantization of the other one. The meaning of the coproduct is that of a non-Abelian group structure on phase-space, describing some kind of non-commutative geometry [13]. The duality means that the quantum algebra of one of the homogeneous spaces is equivalent to the coalgebra (hence geometry) of the other. This possibility of a dual interpretation as quantum mechanics on the one hand and geometry on the other was one of the main ideas introduced in the author's Ph.D. Thesis and we will explore it below for our new examples based on the quantum double. In purely quantum mechanical terms it means that the states of the system also form an algebra and hence a certain symmetry is restored between observables and states. For further details we refer to [9,14–17].

The self-duality of the situation is also connected with other dualities in physics and also with mirror symmetry in string theory. In particular, a natural source of factorizations is provided by the Iwasawa decomposition $G_{\mathbb{C}} = G^*G$ of the complexification of compact semisimple Lie group G into G and a solvable group G^*. The resulting action of G on G^* gives one system with G^* position and G momentum, while the action of G^* on G which arises in the same process gives the dual system with the roles of the position and momentum interchanged. Moreover, the group G^* here is connected at the Lie algebra level with considerations of Manin-triples and the classical double in the theory of classical inverse scattering [1,18]. In [14] we constructed the mutual group actions using the holonomy of a pair of zero-curvature connections. For a concrete example, the Iwasawa decomposition of $\mathrm{SL}(2,\mathbb{C})$ gave rise to a system with position space given by non-concentrically nested spheres, and momentum given by $\mathrm{su}(2)$ (angular momentum). The spheres here are orbits under an action of $\mathrm{SU}(2)$ or $\mathrm{SO}(3)$ on $\mathbb{R}^3$, but an unusual feature was that in order for the quantum algebra of observables to be a Hopf algebra, this action was not the usual rotation but a distorted non-linear one (coming from the Iwasawa decomposition). The non-linearity moreover led to an 'event-horizon' type structure at the plane $z = -1$ in $\mathbb{R}^3$. In this example $\mathrm{SU}(2)^*$ is a solvable group which can be identified with the region $z > -1$. Such 'event horizons' appear to be a characteristic feature of the allowed homogeneous spaces in [9,14–16]. For example, in $1+1$ dimensions the requirement of self-duality forced the metric to be a black-hole type one.

We are going to use much the same ingredients now in our interpretation of the quantum double and its dual. This time the actions will be more standard rotations (in Minkowski spacetime for example) but the novel ingredient now is that the underlying classical system is more naturally a q-deformed geometry rather than an ordinary classical geometry as in the models above. Our constructions are quite general but we concentrate on the doubles $D(U_q(g))$ of the Drinfeld–Jimbo quantum groups $U_q(g)$ [1,19], giving all formulas explicitly for the case where $g = \mathrm{su}_2$.

In this case we take for our q-deformed position observables the $*$-algebra BS_q^3 consisting of $\mathrm{BH}_q(2)$ (the $*$-algebra of 2×2 Hermitian braided-matrices) modulo the condition $\mathrm{BDET} = 1$ where BDET is the braided-determinant. The algebra of 2×2 braided-matrices has been introduced in [20] and has four generators $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$ to be regarded as the 'co-ordinate functions' on the braided space. We equip this now with the $*$-structure $\left(\begin{smallmatrix} a^* & b^*\\ c^* & d^*\end{smallmatrix}\right) = \left(\begin{smallmatrix} a & c\\ b & d\end{smallmatrix}\right)$ appropriate to the Hermitian case, and impose the further determinant condition. Recall that in the undeformed case the Hermitian matrices can be identified with Minkowski space and the determinant with the Lorentz metric. The algebra BS_q^3 is therefore a q-deformation of the co-ordinate functions on the sphere S^3_{Lor} of unit radius in Minkowski space. Meanwhile, the q-deformed momentum of our system is given by the quantum group $U_q(\mathrm{su}_2)$. Recall that in the undeformed situation $\mathrm{SL}(2,\mathbb{C})$ acts on the Hermitian matrices $\mathrm{H}(2)$ (Minkowski space) and preserves the determinant. Its subgroup $\mathrm{SU}(2)$ acts precisely by spatial rotations of Minkowski space and of S^3_{Lor}. In our q-deformed setting the action of $U_q(\mathrm{su}_2)$ corresponds precisely to this action by conjugation, in the form of the quantum adjoint action.

By quantization of this classical q-deformed system we mean a $*$-algebra containing the position observables and momentum (quantum) group in such a way that the action of momentum is implemented in the algebra. This we understand as the construction of a (quantum group) $*$-algebra cross product which is a natural generalization of the Mackey quantization scheme used above [9]. A recent work in which $*$-algebra cross products are understood as quantization is in [21]. In our case the above q-deformed classical system has such a quantization. Moreover, it is a Hopf algebra and is isomorphic as a Hopf algebra to Drinfeld's quantum double $D(U_q(\mathrm{su}_2))$. Full details of this example appear at the end of section 3 below.

Note that q here is considered quite orthogonal to the process of quantization, and need not be related to any physical $\hbar$. Thus in our interpretation we have two independent processes, which mutually commute,

$$\begin{array}{ccc} C(S^3_{\mathrm{Lor}}) \otimes \mathrm{su}_2 & \overset{\text{deformation}}{\rightarrow} & \mathrm{BS}^3_q \otimes U_q(\mathrm{su}_2) \\ \text{quantization} \downarrow & & \downarrow \text{quantization} \\ C(S^3_{\mathrm{Lor}}) \rtimes U(\mathrm{su}_2) & \overset{\text{deformation}}{\rightarrow} & \mathrm{BS}^3_q \rtimes U_q(\mathrm{su}_2). \end{array} \tag{1}$$

The top left here can be viewed as a subset of the classical observables $C(T^*S^3_{\mathrm{Lor}})$ consisting of certain functions on $T^*S^3_{\mathrm{Lor}}$ that are linear in the fiber direction, namely the tensor product of functions on the base S^3_{Lor} and vector fields induced by the action of su_2. This is the natural subset that is quantized in the Mackey scheme (cf. the vertical polarization). The bottom left is the usual Mackey quantization of this system and is isomorphic as an algebra to Drinfeld's double $D(\mathrm{SU}(2))$. We have already explained in [9, example 2.4] that Drinfeld's quantum double of a group or enveloping algebra is a semidirect product and hence forms an example of the general class of models on homogeneous spaces as discussed above. The rest of the diagram is filled in by the constructions in the present paper. Note in this context that the classical situation works just as well with position observables $\mathrm{SU}(2) = S^3$ (in Euclidean space). The action of momentum $\mathrm{SU}(2)$ is again by conjugation, with orbits again spheres. The quantization as an algebra is again Drinfeld's quantum double $C(\mathrm{SU}(2)) \rtimes U(\mathrm{su}_2) \cong D(\mathrm{SU}(2))$. The only difference is that the $*$-structure on the matrix co-ordinate functions is that appropriate to unitary rather than to Hermitian matrices as above. On the other hand the naive q-deformation of this Euclidean system to $\mathrm{SU}_q(2) \rtimes U_q(\mathrm{su}_2)$ is not possible because the quantum adjoint action fails to respect the algebra structure of $\mathrm{SU}_q(2)$ for $q \neq 1$. In fact, I do not know how to q-deform this Euclidean situation and have been forced by this failure into the Minkowski setting and its q-deformation via braided matrices rather than more familiar quantum ones.

One can ask: Why should we be interested in a quantum system whose underlying classical system is more naturally a q-deformed classical geometry? One reason is that we have an extra q parameter to regularise any singularities that appear in the quantum theory (even if, after renormalization, we set $q = 1$) [22]. Related to this it is interesting that the requirement of existence of a q-deformation leads us into the Minkowski setting even if we are interested in the end only in $q = 1$. Another motivation is that at the Planck scale we can expect some feedback between quantization and geometry in the sense that a correct formulation of quantum particles may surely require them to be moving in the background of something other than a usual geometry. Replacing the latter by a q-geometry is one possibility for such models, with q expressing quantum corrections to the background geometry itself.

In general, while the need for some kind of non-commutative or q-geometry has become clear, there remains a shortage of natural examples and physical principles to govern such a geometry (simply asking to q-deform everything still leaves a lot of possibilities). Hence it is significant that the quantum double provides a natural example, the study of which can help develop the subject further. For example, one can study q-deformed differential structures and hope to give a more conventional (but q-deformed) picture of the quantum double as quantizing a q-symplectic or q-Poisson space. One can also try to take a different line and combine the two steps in (1) into a single quantization of a pair of compatible Poisson brackets as in [23]. We will not attempt these steps here, but see the concluding remarks at the end of the paper.

Let us recall now that our original motivation for quantum systems whose algebra of observables are Hopf algebras was an interesting quantum/gravity duality phenomenon implemented by Hopf algebra duality. This is the topic of section 4 where we study the dual of the quantum double. It is also a Hopf algebra, but our result is that it too is a cross product quantization. Here again we depend on the general algebraic theory in [6] but in the context now of $*$-algebras and in more suitable right-handed conventions. The most unusual feature of the result is that this time, for the dual of the quantum double to be a $*$-algebra cross product (as needed for its interpretation as generalized Mackey quantization), the quasitriangular structure should be antireal rather than real. For example, it is the dual of the quantum double of $U_q(\mathrm{sl}(2,\mathbb{R}))$ that has the desired interpretation, rather than of the compact real form $U_q(\mathrm{su}_2)$ above. Thus we have for the dual of the quantum double in this case the interpretation

$$\begin{array}{ccc} C(\mathrm{SL}(2,\mathbb{R}))\otimes \mathrm{su}_2^* & \stackrel{\text{deformation}}{\longrightarrow} & \mathrm{BSL}_q(2,\mathbb{R})\otimes U_q(\mathrm{su}_2^*) \\ \text{quantization}\downarrow & & \downarrow\text{quantization} \\ C(\mathrm{SL}(2,\mathbb{R}))\rtimes U(\mathrm{su}_2^*) & \stackrel{\text{deformation}}{\longrightarrow} & \mathrm{BSL}_q(2,\mathbb{R})\rtimes U_q(\mathrm{su}_2^*). \end{array} \qquad (2)$$

The role of angular momentum is now played by the Drinfeld dual su_2^* as mentioned above. Its q-deformation is by definition $U_q(\mathrm{su}_2^*) = U_q(\mathrm{sl}_2)^*$, i.e., the quantum-group function algebra dual to $U_q(\mathrm{sl}_2)$ but regarded nevertheless as a quantum enveloping algebra. The role of position is again played in the deformed case by the braided group $\mathrm{BSL}_q(2,\mathbb{R})$ which is like the above but with $*$-structure $\begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Full details appear at the end of section 4. The Mackey-type quantization of this system is then the dual Hopf algebra to the quantum double and hence dual to the system (1). This possibility of a dual interpretation is an interesting direction for further work and suggests a genuinely new physical phenomenon. According to the duality principle introduced in [16], this dual system should appear relative to our initial system as physics 'beyond' the Planck scale. This is necessarily a speculative topic but its elaboration remains a long-term motivation for the present work.

We conclude in section 5 with some remarks connecting the constructions here with other points of view in [24,4]. We explain in the last of these that the quantum algebras of observables above can just as easily be understood in a geometric way as principal bundles on quantum homogeneous spaces. When the theory of principal bundles is formulated in the setting of non-commutative algebraic geometry there is not really any difference between semidirect products viewed as quantization and semidirect products viewed as (algebraic) geometry. Hence the double and its dual both have this geometrical interpretation as well as the quantum one above. This is an important principle that is surely relevant to Planck-scale physics where geometry and quantum theory need to be unified.

Throughout this paper we work over a field with involution, which we fix for concreteness to be $\mathbb{C}$. The general results hold for an arbitrary field with involution. Thus our approach to quantization is an algebraic one as explained in detail in [9, section 1.1.1]. The geometrical content is also to be understood in a setting of non-commutative algebraic geometry as indicated above. One can try to place these results explicitly in a C^*-algebra or von Neumann algebra setting though we shall not attempt to do so here. This, and the interpretation of the q-deformed Mackey construction in terms of q-deformed symplectic structures etc. are two directions for further work. The present paper is a necessary first step.

PRELIMINARIES

Here we collect some basic algebraic facts about Hopf algebra cross products and cross coproducts, the quantum double and braided groups.

Recall first that a Hopf algebra is (H, Δ, ϵ, S) where H is an algebra with unit, Δ: $H \to H \otimes H$ (the comultiplication), ϵ: $H \to \mathbb{C}$ (the counit) are algebra maps and S: $H \to H$ (the antipode) plays the role of inverse. An introduction to the axioms is in [25]. We often use the formal sum notation $\Delta h = \sum h_{(1)} \otimes h_{(2)}$ [26]. If H is finite dimensional then $A = H^*$ is also a Hopf algebra, dual to H, with multiplication determined by the comultiplication in H and vice-versa. A similar situation holds in the infinite-dimensional case if we work with a q-adic or other topology as in [1]. An algebraic alternative, which we adopt, is simply to work with dually paired Hopf algebras. Thus $U_q(\mathrm{su}_2)$ is nondegenerately paired with $\mathrm{SU}_q(2)$ etc., in a standard way. For brevity some of our abstract results are proven in the finite-dimensional case: their extension to the dually-paired case is straightforward and verified directly when we come to the relevant examples.

When a Hopf algebra H acts on an algebra B in such a way as to respect its structure in the sense

$$\begin{aligned} h\triangleright(bc) &= \sum(h_{(1)}\triangleright b)(h_{(2)}\triangleright c), \\ h\triangleright 1 &= \epsilon(h)1 \end{aligned} \tag{3}$$

(one says that B is an H-module algebra), one may form a *cross product* algebra $B\rtimes H$. This is just as familiar for group actions. It is built on $B\otimes H$ with product

$$(b\otimes h)(c\otimes g) = \sum b(h_{(1)}\triangleright c)\otimes h_{(2)}g, \qquad b,c\in B,\ h,g\in H. \tag{4}$$

Put another way, $B\rtimes H$ has B,H as subalgebras and cross relations given by $(1\otimes h)(b\otimes 1) = \sum(h_{(1)}\triangleright b\otimes 1)(1\otimes h_{(2)})$. Also, a feature of Hopf algebras is that Hopf algebra constructions also have dual versions obtained by writing the relevant maps as arrows and reversing them. Thus, if B is a coalgebra on which H coacts by $\beta: B\to H\otimes B$ in such a way as to preserve its structure,

$$\begin{aligned} (\mathrm{id}\otimes\Delta_B)\beta(b) &= \sum b_{(1)}{}^{\overline{(1)}}b_{(2)}{}^{\overline{(1)}}\otimes b_{(1)}{}^{\overline{(2)}}\otimes b_{(2)}{}^{\overline{(2)}}, \\ (\mathrm{id}\otimes\epsilon_B)\beta(b) &= 1\epsilon_B(b), \end{aligned} \tag{5}$$

where $\beta(b) = \sum b^{\overline{(1)}}\otimes b^{\overline{(2)}}$ (one says that B is an H-comodule coalgebra), we can form a *cross coproduct coalgebra* $B\rtimes H$. Explicitly, it is built on $B\otimes H$ with coproduct

$$\Delta(b\otimes h) = \sum b_{(1)}\otimes b_{(2)}{}^{\overline{(1)}}h_{(1)}\otimes b_{(2)}{}^{\overline{(2)}}\otimes h_{(2)}. \tag{6}$$

An introduction to these topics is in [25, section 6].

If H is a (say, finite-dimensional) Hopf algebra there is a *quantum double* Hopf algebra $D(H)$ built on $H^*\otimes H$ as follows. The comultiplication, counit and unit are the tensor product ones. So $\Delta(a\otimes h) = \sum a_{(1)}\otimes h_{(1)}\otimes a_{(2)}\otimes h_{(2)}$. The product is twisted,

$$\begin{aligned} &(a\otimes h)(b\otimes g) = \sum b_{(2)}a\otimes h_{(2)}g\langle Sh_{(1)},b_{(1)}\rangle\langle h_{(3)},b_{(3)}\rangle, \\ &h,g\in H;\ a,b\in A = H^*. \end{aligned} \tag{7}$$

This Hopf algebra was introduced by Drinfeld via generators and relations [1]. We use here the abstract form due to the author in [9] and in the conventions of [5]. Note that H and $H^{*\mathrm{op}}$ (H^* with the opposite product) are sub-Hopf algebras. Moreover, the quantum double factorizes into these two factors. The general theory of factorizations of Hopf algebras was introduced in [9]. One shows that for any such factorization, the two factors act on each other and the entire Hopf algebra can be recovered from the factors as a double-cross product. Thus $D(H) \cong H \bowtie H^{*\mathrm{op}}$ where the actions are coadjoint actions [9, example 4.6]. In the infinite-dimensional case we use a formulation in terms of dually paired Hopf algebras along the lines given in [27]. Also, the dual $D(H)^*$ of the quantum double is also a Hopf algebra. It is built on $H\otimes H^*$ as an algebra, with a twisted comultiplication

$$\Delta(h\otimes a) = \sum h_{(2)}\otimes(Sf^b{}_{(1)})a_{(1)}f^b{}_{(3)}\otimes e_b\otimes a_{(2)}\langle f^b{}_{(2)}, h_{(1)}\rangle, \tag{8}$$

where $\{e_b\}$ is a basis of H and $\{f^b\}$ a dual basis.

A Hopf algebra H is quasitriangular (a quantum group of enveloping algebra type) if there is an invertible element $\mathcal{R}\in H\otimes H$ obeying the axioms of Drinfeld [1],

$$(\Delta\otimes\mathrm{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23},\quad (\mathrm{id}\otimes\Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12},\quad \Delta^{\mathrm{op}} = \mathcal{R}(\Delta\ )\mathcal{R}^{-1} \tag{9}$$

where Δ^{op} denotes the opposite comultiplication. We define $Q=\mathcal{R}_{21}\mathcal{R}_{12}$ to be the *quantum Killing form*. A quasitriangular Hopf algebra H is called factorizable [28] if Q is non-degenerate in the sense that the set $\{(\phi\otimes\mathrm{id})(Q)|\phi\in H^*\}\subseteq H$ coincides with all of H. The quantum double of any Hopf algebra is quasitriangular and indeed, factorizable, as are many familiar quantum groups such as $U_q(g)$ (working with suitable topological completions or with a designated dually-paired Hopf algebra in the role of H^*). Clearly, the dual concept of a quasitriangular Hopf algebra is a Hopf algebra A equipped with a convolution-invertible map $\mathcal{R}$: $A\otimes A\to\mathbb{C}$ obeying some obvious axioms dual to (9), namely

$$\begin{aligned}\mathcal{R}(ab\otimes c) &= \sum\mathcal{R}(a\otimes c_{(1)})\mathcal{R}(b\otimes c_{(2)}),\\ \mathcal{R}(a\otimes bc) &= \sum\mathcal{R}(a_{(1)}\otimes c)\mathcal{R}(a_{(2)}\otimes b),\end{aligned} \tag{10}$$

$$\sum b_{(1)}a_{(1)}\mathcal{R}(a_{(2)}\otimes b_{(2)}) = \sum\mathcal{R}(a_{(1)}\otimes b_{(1)})a_{(2)}b_{(2)} \tag{11}$$

for all a, b, $c\in A$. The dual quantum groups (quantum groups of function algebra type) are like this. Finally, the axioms of a (dual) quasitriangular Hopf algebra are such that its category of (co)modules forms a braided or quasitensor category. If V, W are H-modules, the braided-transposition or braiding is

$$\Psi_{V,W}\colon V\otimes W\to W\otimes V,\quad \Psi_{V,W}(v\otimes w) = \tau(\mathcal{R}\triangleright(v\otimes w)), \tag{12}$$

where $\mathcal{R}\in H\otimes H$ acts on $V\otimes W$. It is an intertwiner for the action of H and has properties similar to those of the usual transposition τ, except that $\Psi_{V,W}$ and $\Psi_{W,V}^{-1}$ need not coincide (they are usually represented as distinct braids). Similarly for the category of comodules in the dual quasitriangular case.

Next we need the notion of an algebra *living in* a braided category. We will be concerned here only with the braided categories of representations of a quantum group as just explained. In this case, an algebra in the category means an ordinary algebra B, on which the quantum group acts, in such a way that the algebra and unit maps are covariant. The tensor product action on $B\otimes B$ is given by the action of $\Delta(H)$. Hence the necessary condition is just that B is an H-module algebra as in (3). We saw above that this led to cross products. On the other hand our new category-theoretical view of H-module algebras is also very useful and leads to the notion of braided tensor products of H-module algebras when H is a quantum group [10]. The idea is that if B, C are two algebras living in a braided category, then $B\otimes C$ has an algebra structure, which we denote $B\underline{\otimes} C$, also living in the category. In our present case this is

$$(b\otimes c)(d\otimes e) = b\Psi_{C,B}(c\otimes d)e = \sum b(\mathcal{R}^{(2)}\triangleright d)\otimes(\mathcal{R}^{(1)}\triangleright c)e, \tag{13}$$

where the second expression uses (12). One can check that this is associative.

This is one of the fundamental constructions behind the author's theory of braided groups. For a braided group B is a kind of Hopf algebra in which the braided-coproduct is taken as an algebra homomorphism $\underline{\Delta}$: $B\to B\underline{\otimes} B$ with the braided tensor product structure $B\underline{\otimes} B$ from (13). The structure $B=\underline{H}$ needed above is an example. Explicitly,

$$\underline{H} = \begin{cases} H & \text{as algebra} \\ \underline{\Delta} & \text{modified comultiplication} \end{cases} \qquad \underline{\Delta} b = \sum b_{(1)}S\mathcal{R}^{(2)}\otimes\mathcal{R}^{(1)}\triangleright b_{(2)}, \tag{14}$$

where $\triangleright$ is the quantum adjoint action $h\triangleright b = \sum h_{(1)}bSh_{(2)}$. There is also a modified antipode. Thus $\underline{H}$ is not an ordinary Hopf algebra but instead, $\underline{\Delta}: \underline{H}\to \underline{H}\underline{\otimes}\underline{H}$ is an algebra homomorphism provided $\underline{H}$ is treated with braid statistics as in (13). It is the braided group associated to H (of enveloping algebra type) and lives in the braided category of H-modules by the quantum adjoint action.

There is also a braided group of function algebra type associated to a Hopf algebra A dual to H. It is

$$\underline{A} = \begin{cases} A & \text{as coalgebra} \\ \underline{\cdot} & \text{modified product} \end{cases} \qquad a\,\underline{\cdot}\,b = \sum a_{(2)}b_{(2)}\mathcal{R}((Sa_{(1)})a_{(3)}\otimes Sb_{(1)}). \tag{15}$$

It also has a modified antipode. This $\underline{A}$ lives in the braided category of left H-modules (or right A-comodules) by the coadjoint action

$$h\triangleright a = \sum\langle h, a_{(2)}\rangle(Sa_{(1)})a_{(3)}; \quad a\in A,\ h\in H \tag{16}$$

(or right adjoint coaction $\beta(a) = \sum a_{(2)}\otimes(Sa_{(1)})a_{(3)}$). When H is factorizable and has dual A, then one finds remarkably that $\underline{H}\cong\underline{A}$ as braided groups [6]. For a systematic introduction to braided groups one can see [29].

Armed with these various ingredient we have shown in [6] cf. [5] that

$$D(H)\cong \underline{A}\rtimes H \tag{17}$$

when H is quasitriangular and $\underline{A}$ is the associated braided group of function algebra type. The motivation in [6] was in relation to the role of $D(H)$ as a complexification of H. Our new goal in the present paper is to give the interpretation of this semidirect product theorem as quantum mechanics on the 'braided-space' underlying $\underline{A}$.

2. $*$-Structures on braided groups

If we are to develop a quantum-mechanical interpretation of the quantum double, we are going to have to work with $*$-algebras. The $*$-algebra structure on a quantum algebra of observables determines such things as positivity of states, the possibility of Hilbert space representations etc. Such $*$-structures on Hopf algebras are well-known since the pioneering work of [30], but the theory of $*$-structures on braided groups (which we will need in later sections) has not previously been developed. This is our goal in the present section. After this, one may construct norms and obtain operator versions of the various Hopf algebras and braided groups, along established lines such as in [30].

Let us recall that a Hopf $*$-algebra is a Hopf algebra where H is itself a $*$-algebra and Δ, ϵ are $*$-algebra homomorphisms. In addition, one requires $(S \circ *)^2 = \mathrm{id}$ [31]. In this case the dual, A, is also a $*$-algebra. Its $*$-structure is related to that of H via

$$\langle h^*, a\rangle = \overline{\langle h, (Sa)^*\rangle}, \qquad \forall h \in H,\ a \in A. \tag{18}$$

Now, when H is a $*$-quantum group (a quasitriangular Hopf $*$-algebra) it is natural to impose either of the two conditions

$$\mathcal{R}^{*\otimes *} = \mathcal{R}_{21}\ \text{(real)}, \qquad \mathcal{R}^{*\otimes *} = \mathcal{R}^{-1}\ \text{(antireal)}. \tag{19}$$

This is because H^{op} has two natural quasitriangular structures, namely $\mathcal{R}_{21}$ and $\mathcal{R}^{-1}$ and so one would expect that $*$: $H \to H^{\mathrm{op}}$ should map $\mathcal{R}$ to one or other of them. The first of these cases has been noted already in [12] and as well as in [21] in the context of quantum mechanics on the quantum sphere.

Example 2.1. *Let H be a Hopf $*$-algebra, then so is $D(H)$ with $(a \otimes 1)^* = (S^2 a)^* \otimes 1$, $(1 \otimes h)^* = (1 \otimes h^*)$. Its canonical quasitriangular structure is antireal.*

Proof. For simplicity we consider only the finite-dimensional case, but see [27]. We first make $H^{*\mathrm{op}}$ (with the reversed product of H^*) into a Hopf $*$-algebra with $*^{\mathrm{op}} = S^{-2} \circ * = * \circ S^2$ and antipode S^{-1} (any even power in place of S^2 will do here for $*^{\mathrm{op}}$, but this is the one that we need in our present conventions for the quantum double). Since the quantum double is generated by these Hopf algebras there is then a unique possibility for its $*$-structure such that the inclusions are $*$-algebra maps as stated. On a general element we take

$(a\otimes h)^* = ((a\otimes 1)(1\otimes h))^* = (1\otimes h^*)((S^2a)^*\otimes 1)$. Putting this into (7) and using (18) we see that this is indeed a $*$-algebra structure on $D(H)$:

$$\begin{aligned}
&[(a\otimes h)(b\otimes g)]^* = \\
&\quad \sum (1\otimes g^*)(1\otimes h_{(2)}{}^*)((S^2b_{(2)})^*\otimes 1)((S^2a)^*\otimes 1) \\
&\qquad \times \overline{\langle Sh_{(1)}, b_{(1)}\rangle\langle h_{(3)}, b_{(3)}\rangle} \\
&\quad = \sum (1\otimes g^*)(1\otimes h_{(2)}{}^*)((S^2b_{(2)})^*\otimes 1)((S^2a)^*\otimes 1) \\
&\qquad \times \langle (Sh_{(1)})^*, (Sb_{(1)})^*\rangle\langle h_{(3)}{}^*, (Sb_{(3)})^*\rangle \\
&\quad = \sum (1\otimes g^*)(1\otimes h^*{}_{(2)})((S^2b)^*{}_{(2)}\otimes 1)((S^2a)^*\otimes 1) \\
&\qquad \times \langle h^*{}_{(1)}, (S^2b)^*{}_{(1)}\rangle\langle h^*{}_{(3)}, S(S^2b)^*{}_{(3)}\rangle \\
&\quad = \sum (1\otimes g^*)((S^2b)^*{}_{(2)(2)}\otimes h^*{}_{(2)(2)})((S^2a)^*\otimes 1) \\
&\qquad \times \langle Sh^*{}_{(2)(1)}, (S^2b)^*{}_{(2)(1)}\rangle \\
&\qquad \times \langle h^*{}_{(2)(3)}, (S^2b)^*{}_{(2)(3)}\rangle\langle h^*{}_{(1)}, (S^2b)^*{}_{(1)}\rangle\langle h^*{}_{(3)}, S(S^2b)^*{}_{(3)}\rangle \\
&\quad = (1\otimes g^*)((S^2b)^*\otimes h^*)((S^2a)^*\otimes 1) \\
&\quad = (1\otimes g^*)((S^2b)^*\otimes 1)(1\otimes h^*)((S^2a)^*\otimes 1) = (b\otimes g)^*(a\otimes h)^*.
\end{aligned}$$

For the fifth equality we used the duality between H^* and H and the antipode axioms. Moreover, $(S\circ *)(a\otimes h) = S[(1\otimes h^*)((S^2a)^*\otimes 1)] = S^{-3}(a^*)\otimes S(h^*)$ so that $(S\circ *)^2 = \mathrm{id}$ on $D(H)$. Since the coalgebra of the quantum double is the tensor product one, it it equally clear that it commutes with $*$ as it should.

The standard quasitriangular structure [1] in our conventions is given by $\mathcal{R} = \sum_a (f^a\otimes 1)\otimes(1\otimes e_a)$ where $\{e_a\}$ is a basis of H and $\{f^a\}$ a dual basis. Hence $\mathcal{R}^{*\otimes *} = \sum_a (S^{-2}(f^{a*})\otimes 1)\otimes(1\otimes e_a{}^*) = \sum (S^{-1}f'^a\otimes 1)\otimes(1\otimes e'_a) = (S\otimes\mathrm{id})\mathcal{R} = \mathcal{R}^{-1}$, where $e'_a = e_a{}^*$ is a new basis with dual basis $f'^a = S^{-1}(f^{a*})$ according to (18).

Example 2.2. *Let $H = U_q(\mathrm{su}_2)$ equipped with its usual $*$-structure $H^* = H$, $X_\pm^* = X_\mp$ in conventions where $[X_+, X_-] = (q^H - q^{-H})/(q-q^{-1})$ with $q = \mathrm{e}^{t/2}$ real. Its standard quasitriangular structure over $\mathbb{C}[[t]]$ is real.*

Proof. This fact has already been observed in [12,21] but we include the proof here for completeness. The standard expression for $\mathcal{R}$ for $U_q(\mathrm{su}_2)$ is [32]

$$\mathcal{R} = q^{H\otimes H/2} e_q^{(1-q^{-2})q^{H/2}X_+\otimes q^{-H/2}X_-} = e_q^{(1-q^{-2})X_+q^{-H/2}\otimes X_-q^{H/2}} q^{H\otimes H/2}$$

where e_q is a q-exponential $e_q^x = \sum_{m=0}^{\infty} x^m/[[m]]!$ and $[[m]] = \frac{1-q^{-2m}}{1-q^{-2}}$. The second expression coincides using the relations in the algebra. Starting with the first expression for $\mathcal{R}$ we then compute

$$\begin{aligned}\mathcal{R}^{*\otimes *} &= e_{q^*}^{(1-q^{-2})^*(q^{H/2}X_+)^*\otimes(q^{-H/2}X_-)^*}(q^{H\otimes H/2})^* \\ &= e_q^{(1-q^{-2})X_-q^{H/2}\otimes X_+q^{-H/2}}q^{H\otimes H/2} = \mathcal{R}_{21}.\end{aligned}$$

Example 2.3. *Let $H = U_q(\mathrm{sl}(2,\mathbb{R}))$ with its usual $*$-structure $H^* = -H$, $X_\pm^* = -X_\pm$ and with $|q| = 1$. Its standard quasitriangular structure over $\mathbb{C}[[t]]$ is antireal.*

Proof. We compute

$$\begin{aligned}\mathcal{R}^{*\otimes *} &= e_{q^*}^{(1-q^{-2})^*(q^{H/2}X_+)^*\otimes(q^{-H/2}X_-)^*}(q^{H\otimes H/2})^* \\ &= e_{q^{-1}}^{(1-q^2)X_+q^{H/2}\otimes X_-q^{-H/2}}q^{-H\otimes H/2} \\ &= e_{q^{-1}}^{-(1-q^{-2})q^{H/2}X_+\otimes q^{-H/2}X_-}q^{-H\otimes H/2} \\ &= \left(e_q^{(1-q^{-2})q^{H/2}X_+\otimes q^{-H/2}X_-}\right)^{-1}q^{-H\otimes H/2} = \mathcal{R}^{-1}\end{aligned}$$

by the relations in the algebra (to organise the exponent of $e_{q^{-1}}$) and the observation $(e_q^{\lambda x})^{-1} = e_{q^{-1}}^{-\lambda x}$ with the operator $q^{H/2}X_+\otimes q^{-H/2}X_-$ formally in the role of x. Another proof is to compute $\mathcal{R}^{-1} = (S\otimes\mathrm{id})(\mathcal{R})$ directly from the formula for $\mathcal{R}$ and compare with the third expression for $\mathcal{R}^{*\otimes *}$ above.

So both real and antireal quasitriangular Hopf $*$-algebras exist (in abundance). We are interested mainly in the real case for the following purpose:

Definition 2.4. Let H be a real-quasitriangular Hopf $*$-algebra and $\mathcal{C}$ its braided category of representations. A *real Hopf* $\underline{*}$-algebra B living in this category is a Hopf algebra in the category (so $\underline{\Delta}: B \to B\underline{\otimes}B$ to the braided tensor product algebra) such that

(a) $\underline{*}: B \to B$ is a $*$-algebra structure on B (so it is antilinear, $(\underline{*})^2 = \mathrm{id}$ and $(bc)^{\underline{*}} = c^{\underline{*}}b^{\underline{*}}$).

(b) $\underline{\Delta}\circ\underline{*} = \tau\circ(\underline{*}\otimes\underline{*})\circ\underline{\Delta}$ where τ is usual transposition, and $\underline{\epsilon}\circ * = \bar{\ }\underline{\epsilon}$.

(c) $\underline{S}\circ\underline{*} = \underline{*}\circ\underline{S}$ where $\underline{S}$ is the braided-antipode.

(d) The action of H on B obeys $(h\triangleright b)^{\underline{*}} = (Sh)^*\triangleright b^{\underline{*}}$ for $h\in H$, $b\in B$.

Although the most general axiom system for $*$-structures on Hopf algebras in braided categories is not known, the notion introduced here is relevant for our quantum-mechanical purposes and we will see next that it does hold for the braided groups $\underline{H}$, $\underline{A}$ in the preliminaries. Both the product and coproduct are skew with respect to $\underline{*}$. One can easily see that given (a), (d) in definition 2.2 the braided tensor product algebra $B\underline{\otimes}B$ has the $*$-algebra structure cf. [21]

$$\underline{*}_{B\underline{\otimes}B} = \tau \circ (\underline{*} \otimes \underline{*})$$

when $\mathcal{R}$ is real. In these terms the relevant part of condition (b) is equivalent to

(b)′ $\underline{\Delta} \circ \underline{*} = \underline{*}_{B\underline{\otimes}B} \circ \underline{\Delta}$.

Proposition 2.5. *If H is a real-quasitriangular Hopf $*$-algebra then the braided group $B = \underline{H}$ with $\underline{*} = *$ is a real Hopf $*$-algebra in the braided category of H-modules.*

Proof. The quantum adjoint action of H on $\underline{H}$ (which we identify with H as a $*$-algebra) always obeys condition (d) for any Hopf algebra H since,

$$(h \triangleright b)^* = \sum (h_{(1)} b S h_{(2)})^* = \sum (S h_{(2)})^* b^* h_{(1)}{}^* = \sum (S^{-1} h^*{}_{(2)}) b^* h^*{}_{(1)}$$
$$= \sum (S^{-1} h^*{}_{(2)}) b^* S S^{-1} h^*{}_{(1)} = \sum (Sh)^*{}_{(1)} b^* S (Sh)^*{}_{(2)} = (Sh)^* \triangleright b^*.$$

For condition (b) we compute

$$\begin{aligned}
(* \otimes *)\underline{\Delta} b &= \sum (b_{(1)} S\mathcal{R}^{(2)})^* \otimes (\mathcal{R}^{(1)} \triangleright b_{(2)})^* \\
&= \sum (S\mathcal{R}^{(2)})^* b_{(1)}{}^* \otimes (S\mathcal{R}^{(1)})^* \triangleright b_{(2)}{}^* \\
&= \sum \mathcal{R}^{(1)} b^*{}_{(1)} \otimes \mathcal{R}^{(2)} \triangleright b^*{}_{(2)} \\
&= \sum \mathcal{R}^{(1)} b^*{}_{(1)} \otimes \mathcal{R}^{(2)}{}_{(1)} b^*{}_{(2)} S\mathcal{R}^{(2)}{}_{(2)} \\
&= \sum \mathcal{R}^{(1)} \mathcal{R}'^{(1)} b^*{}_{(1)} \otimes \mathcal{R}'^{(2)} b^*{}_{(2)} S\mathcal{R}^{(2)} \\
&= \sum \mathcal{R}^{(1)} b^*{}_{(2)} S\mathcal{R}'^{(1)} \otimes b^*{}_{(1)} S\mathcal{R}'^{(2)} S\mathcal{R}^{(2)} \\
&= \sum \mathcal{R}^{(1)}{}_{(1)} b^*{}_{(2)} S\mathcal{R}^{(1)}{}_{(2)} \otimes b^*{}_{(1)} S\mathcal{R}^{(2)} \\
&= \tau \circ \underline{\Delta} b^*.
\end{aligned}$$

Here $\mathcal{R}'$ is a second copy of $\mathcal{R}$ and we used the axioms (9) freely as well as $(S \otimes S)(\mathcal{R}) = \mathcal{R}$. The braided-counit $\underline{\epsilon}$ coincides with the usual one, so like the algebra structure, it automatically has the required properties. For the braided-antipode we compute from the expression [10] $\underline{S} b = \sum \mathcal{R}^{(2)} u^{-1} S(\mathcal{R}^{(1)} b)$ where $u^{-1} = \sum \mathcal{R}^{(2)} S^2 \mathcal{R}^{(1)}$,

$$\begin{aligned}
* \circ \underline{S} b &= \sum (S^{-1}(b^* \mathcal{R}^{(1)*})) u^{-1} \mathcal{R}^{(2)*} = \sum (S^{-1}(b^* \mathcal{R}^{(2)})) u^{-1} \mathcal{R}^{(1)} \\
&= \sum (S^{-1} \mathcal{R}^{(2)}) u^{-1} (S b^*) S S^{-1} \mathcal{R}^{(1)} = \sum \mathcal{R}^{(2)} u^{-1} S(\mathcal{R}^{(1)} b^*) = \underline{S} \circ * b.
\end{aligned}$$

Here $u^{-1*} = \sum (S^{-2} \mathcal{R}^{(1)*}) \mathcal{R}^{(2)*} = S^{-2}(u^{-1}) = u^{-1}$ when $\mathcal{R}$ is real.

Proposition 2.6. *Let A be the dual of a real-quasitriangular Hopf $*$-algebra H. The braided group $B = \underline{A}$ with $\underline{*} = * \circ S$ is a real Hopf $*$-algebra in the braided category of H-modules. Similarly when A is dual quasitriangular and $\underline{A}$ lives in the braided category of A-comodules.*

Proof. First we verify that $\underline{A}$ as an H-module under (16) obeys the required condition (d) (for any Hopf algebra H with dual A). Indeed,

$$\begin{aligned}
(h \triangleright a)^{\underline{*}} &= * \circ S \sum a_{(2)} \langle h, (Sa_{(1)}) S^{-1} Sa_{(3)} \rangle \\
&= \sum (Sa)_{(2)}{}^{*} \overline{\langle h, (Sa)_{(3)} S^{-1} (Sa)_{(1)} \rangle} \\
&= \sum (Sa)_{(2)}{}^{*} \langle (Sh)^{*}, (S(Sa)_{(1)}{}^{*})(Sa)_{(3)}{}^{*} \rangle \\
&= \sum a^{\underline{*}}{}_{(2)} \langle (Sh)^{*}, (Sa^{\underline{*}}{}_{(1)}) a^{\underline{*}}{}_{(3)} \rangle \\
&= (Sh)^{*} \triangleright a^{\underline{*}}.
\end{aligned}$$

Next, the quasitriangular structure on H induces a dual quasitriangular structure $\mathcal{R}(a \otimes b) = \langle \mathcal{R}, a \otimes b \rangle$. From (18) and $(S \otimes S)(\mathcal{R}) = \mathcal{R}$, one finds that the reality condition in these dual terms is

$$\mathcal{R}(a^{*} \otimes b^{*}) = \overline{\mathcal{R}(b \otimes a)}. \tag{20}$$

The structure of $\underline{A}$ was recalled in terms of this $\mathcal{R}$ in the preliminaries. We have

$$\begin{aligned}
(a \underline{\cdot} b)^{\underline{*}} &= * \circ S(a \underline{\cdot} b) = \sum (Sa_{(2)})^{*} (Sb_{(2)})^{*} \overline{\mathcal{R}((Sa_{(1)}) a_{(3)} \otimes Sb_{(1)})} \\
&= \sum a_{(2)}{}^{\underline{*}} b_{(2)}{}^{\underline{*}} \mathcal{R}((Sb_{(1)})^{*} \otimes (S(Sa_{(3)})^{*})(Sa_{(1)})^{*}) \\
&= \sum a_{(2)}{}^{\underline{*}} b_{(2)}{}^{\underline{*}} \mathcal{R}(b_{(1)}{}^{\underline{*}} \otimes (Sa_{(3)}{}^{\underline{*}}) a_{(1)}{}^{\underline{*}}) \\
&= \sum a^{\underline{*}}{}_{(2)} b^{\underline{*}}{}_{(1)} \mathcal{R}(b^{\underline{*}}{}_{(2)} \otimes (Sa^{\underline{*}}{}_{(1)}) a^{\underline{*}}{}_{(3)}) \\
&= \sum a^{\underline{*}}{}_{(2)} b^{\underline{*}}{}_{(1)} \mathcal{R}(b^{\underline{*}}{}_{(2)} \otimes a^{\underline{*}}{}_{(3)}) \mathcal{R}(b^{\underline{*}}{}_{(3)} \otimes Sa^{\underline{*}}{}_{(1)}) \\
&= \sum \mathcal{R}(b^{\underline{*}}{}_{(1)} \otimes a^{\underline{*}}{}_{(2)}) b^{\underline{*}}{}_{(2)} a^{\underline{*}}{}_{(3)} \mathcal{R}(b^{\underline{*}}{}_{(3)} \otimes Sa^{\underline{*}}{}_{(1)}) \\
&= \sum b^{\underline{*}}{}_{(2)} a^{\underline{*}}{}_{(2)} \mathcal{R}((Sb^{\underline{*}}{}_{(1)}) b^{\underline{*}}{}_{(3)} \otimes Sa^{\underline{*}}{}_{(1)}) \\
&= b^{\underline{*}} \underline{\cdot} a^{\underline{*}}
\end{aligned}$$

using the properties (10), (11) of $\mathcal{R}$. It is evident that $(\underline{*})^2 = \mathrm{id}$ and condition (b) hold since A is a Hopf $*$-algebra.

Clearly, the proof of the preceding proposition holds for any real-dual quasitriangular Hopf $*$-algebra $(A, \mathcal{R}\colon A \otimes A \to \mathbb{C})$, with $\underline{A}$ living in the category of right A-comodules by the right adjoint coaction (this is slightly more general than saying that A is dual to some H).

Proposition 2.7. *In the factorizable case of proposition 2.6, the isomorphism Q: $\underline{A} \cong \underline{H}$ due to the quantum Killing form is a $*$-isomorphism.*

Proof. The map here is $a \in \underline{A}$ maps to $Q(a) = (a \otimes \mathrm{id})(Q)$ where $Q = \mathcal{R}_{21}\mathcal{R}_{12}$. In the real case we have $Q^{*\otimes *} = Q$ and hence

$$Q(a^{\underline{*}}) = \sum \langle (Sa)^*, Q^{(1)} \rangle Q^{(2)} = \sum \overline{\langle a, Q^{(1)*} \rangle} Q^{(2)**} = (Q(a))^*.$$

This homomorphism Q is a homomorphism of braided groups [6], and we see now that as such it is a $*$-homomorphism. In the factorizable case it is, by definition an isomorphism.

The process of transmutation of a dual quantum group into a braided one works also at the level of matrix bialgebras. The braided version of the FRT bialgebra $A(R)$ [33] is the braided matrices $B(R)$ introduced in [20]. These have matrix generators $u^i{}_j$ with relations and braided-coproduct

$$R_{21}\mathbf{u}_1 R \mathbf{u}_2 = \mathbf{u}_2 R_{21} \mathbf{u}_1 R, \quad \underline{\Delta} u = u \underline{\otimes} u, \quad \underline{\epsilon} u^i{}_j = \delta^i{}_j. \tag{21}$$

The relations that arise here have been noted in various other contexts also.

Corollary 2.8. *Let $R \in M_n \otimes M_n$ be a matrix solution of the QYBE of real-type in the sense*

$$\overline{R^i{}_j{}^k{}_l} = R^l{}_k{}^j{}_i, \quad \text{i.e.,} \quad \overline{R}^{t\otimes t} = R_{21}.$$

Then

$$u^i{}_j{}^{\underline{*}} = u^j{}_i$$

makes $B(R)$ a real-braided matrix $*$-bialgebra in the sense of (a),(b),(d) of definition 2.4.

Proof. We first motivate the definitions. Thus, consider the dual quantum group A obtained as a quotient of the FRT bialgebra $A(R)$ with a matrix of generators $t^i{}_j$ and relations $R\mathbf{t}_1\mathbf{t}_2 = \mathbf{t}_2\mathbf{t}_1 R$ as in [33]. It is easy to see that if R is of real-type then the compact matrix-pseudogroup $*$-structure $t^i{}_j{}^* = St^j{}_i$ as in [31] is always compatible with the FRT relations – we assume that it descends to the Hopf algebra level of A also. Next, we have shown in [25, section 3] (in some form) that $A(R)$ is indeed dual quasitriangular, i.e. there exists $\mathcal{R}$: $A(R) \otimes A(R) \to \mathbb{C}$ obeying (10), (11). It is $\mathcal{R}(t^i{}_j \otimes t^k{}_l) = R^i{}_j{}^k{}_l$ extended according to (10), and we assume that it descends to $\mathcal{R} : A \otimes A \to \mathbb{C}$. That this $\mathcal{R}$ is real in the sense above corresponds to

$$\begin{aligned}\overline{R^i{}_j{}^k{}_l} &= \overline{\mathcal{R}(t^i{}_j \otimes t^k{}_l)} = \mathcal{R}(t^k{}_l{}^* \otimes t^i{}_j{}^*) = \mathcal{R}(St^l{}_k \otimes St^j{}_i) = \mathcal{R}(t^l{}_k \otimes t^j{}_i) \\ &= R^l{}_k{}^j{}_i.\end{aligned}$$

This is the reason for the condition stated. Assuming that R is of real-type we have for the corresponding $\underline{A}$ as a quotient of $B(R)$ the $*$-structure $\underline{*} = * \circ S =$

$S^{-1}\circ *$ from proposition 2.6. Here the generators of $B(R)$ are identified with those of $A(R)$, $\mathbf{u}=\mathbf{t}$ (but not their products). Hence we have $u^i{}_j{}^{\underline{*}}=S^{-1}t^i{}_j{}^*=t^j{}_i=u^j{}_i$. This is the reason for the definitions.

More generally, we adopt this definition of $\underline{*}$ at the level of the braided matrices and verify directly that $B(R)$ obeys (a),(b) in definition 2.4 as a bialgebra in a braided category. Thus, to see that $\underline{*}$ defines a $*$-algebra structure on $B(R)$, we write out its defining relations

$$\begin{aligned}(R_{21}\mathbf{u}_1R_{12}\mathbf{u}_2)^i{}_j{}^k{}_l &\equiv R^k{}_a{}^i{}_bu^b{}_cR^c{}_j{}^a{}_du^d{}_l\\ &= u^k{}_aR^a{}_b{}^i{}_cu^c{}_dR^d{}_j{}^b{}_l\equiv(\mathbf{u}_2R_{21}\mathbf{u}_1R_{12})^i{}_j{}^k{}_l.\end{aligned}$$

Applying $\underline{*}$ to this and using that R is of real-type, we obtain the same relations in the form

$$\begin{aligned}(\mathbf{u}_2R_{21}\mathbf{u}_1R_{12})^j{}_i{}^l{}_k &= u^l{}_dR^d{}_a{}^j{}_cu^c{}_bR^b{}_i{}^a{}_k\\ &= R^l{}_b{}^j{}_du^d{}_cR^c{}_i{}^b{}_au^a{}_k=(R_{21}\mathbf{u}_1R_{12}\mathbf{u}_2)^j{}_i{}^l{}_k.\end{aligned}$$

This means that $\underline{*}$ indeed extends to $B(R)$ as a $*$-algebra structure. Finally, it obeys

$$\underline{\Delta}(u^i{}_j{}^{\underline{*}})=\underline{\Delta}u^j{}_i=u^j{}_k\otimes u^k{}_i=u^k{}_j{}^{\underline{*}}\otimes u^i{}_k{}^{\underline{*}}=\tau(\underline{*}\otimes\underline{*})\underline{\Delta}u^i{}_j.$$

Finally, the condition (d) makes sense for action of a Hopf algebra $U(R)$ dual to A (and can be checked as such). Alternatively, it can more directly be verified in a corresponding form for the right coaction of A, which takes the form $\beta(u^i{}_j)=u^a{}_b\otimes(St^i{}_a)t^b{}_j$.

If $A=G_q$ is the quantum group obtained from $A(R)$ for the quantum function algebras associated to the standard semisimple Lie algebras as in [33] (such as $\mathrm{SL}_q(n)$) then the corresponding $\underline{A}=\mathrm{BG}_q$ is the braided group of function algebra type. It is a quotient of the $B(R)$ modulo suitable determinant-type relations. In this case there is a braided-antipode and $\underline{*}$ descends to a real-matrix braided group structure on BG_q. This follows from proposition 2.6 as explained in the preceding proof. For example, the braided group $\mathrm{BSL}_q(2)$ computed in [20] now becomes a real $*$-braided group $\mathrm{BSU}_q(2)$ with the $*$-structure above. Bearing in mind that the braided groups of function algebra type are in a certain sense braided-commutative, we see that we can think of $\mathrm{BSU}_q(2)$ as the 'ring of functions' on some kind of braided 3-sphere. On the other hand, because the transmuted $*$-structure is Hermitian this S^3 has become a Lorentzian one rather than the Euclidean one that we began with in the form of $\mathrm{SU}_q(2)$. The motivation for these interpretations is the well-known theorem of Gelfand and Naimark that any commutative C^*-algebra is the functions on some locally compact space. Finally, a corollary of proposition 2.7 is

Corollary 2.9. *Let $U_q(g)$ be a standard quantum group in FRT form with generators $L = l^+Sl^-$. If the corresponding R-matrix is of real-type then*

$$L^i{}_j{}^* = L^j{}_i$$

defines a real-quasitriangular Hopf $*$-algebra structure on $U_q(g)$. This is the case for the standard compact real form of $U_q(g)$ at real q.

Proof. From proposition 2.7 we have $BU_q(g) \cong \mathrm{BG}_q$ as $*$-braided groups. The generator corresponding to u in the isomorphism is $L = l^+Sl^-$ as explained in [6], giving the stated $*$-structure for $BU_q(g)$. But this coincides with $U_q(g)$ as an algebra. A short computation shows that the standard R-matrices as used in [33] are of real type if q is real and that this gives the standard compact $U_q(g)$ $*$-structure. It is also known from general category-theoretical grounds that the standard compact $*$-structures on $U_q(g)$ are real-quasitriangular [12].

Note that even for non-standard R of real type, the definition

$$l^{\pm i}{}_j{}^* = Sl^{\mp j}{}_i \tag{22}$$

is compatible with the relations of the bialgebra $\widetilde{U}(R)$ in [25, section 3] and if it descends to the Hopf algebra quotient $U(R)$, will define a Hopf $*$-algebra structure with $L^* = L^t$ as above. Here the $*$-structure on $U_q(g)$ or $U(R)$ forms a Hopf $*$-algebra. It specifies a real form of the quantum group as a symmetry (such as angular momentum).

3. $D(H)$ as a quantum algebra of observables

In this section we apply the algebraic results of the preceding section to give a quantum-mechanical picture of the quantum double. We begin by recalling the relevant definition of quantization according to the cross product construction of Mackey. In the classical case it can be understood in more conventional terms as quantization of Poisson brackets etc., but this need not concern us now. It has an obvious generalization to deal with q-groups on q-spaces which we will then use.

Thus, let X be a manifold and G a Lie group with Lie algebra g, acting on X. We consider the quantization of a particle on X moving (on orbits) according to the action of G (in nice cases there is a metric on X such that this motion is geodesic motion). The position observables are functions $C(X)$ (for example, the C^*-algebra of functions vanishing at infinity when X is locally compact), while the momentum observables are elements ξ of g. They are constants of the motion. Quantization of the system means to find a $*$-algebra $\mathcal{A}$ (usually a $*$-subalgebra of operators on a Hilbert space) containing the position and momentum observables as $*$-subalgebras such that

$$\widehat{e^{\tau\xi}}\,\widehat{f}\,\widehat{e^{-\tau\xi}} = \widehat{\alpha_{e^{\tau\xi}}(f)}, \qquad \alpha_{e^{\tau\xi}}(f)(x) = f(\alpha_{e^{-\tau\xi}}(x)) \tag{23}$$

where α denotes the action on X and also the induced action on $f \in C(X)$, and $\widehat{\ }$ denotes the embeddings of the position and momentum observables in $\mathcal{A}$ (the quantization maps). This requirement is just a co-ordinate-free version of the usual Heisenberg commutation relations, for it says that conjugation by the exponentiated momentum in the quantum algebra implements translation on the underlying position space.

On the other hand, there is a universal solution to this quantization problem. It is the cross product algebra $C(X) \rtimes G$ by α. Here, for convenience, we have shifted from the Lie algebra g (or its enveloping algebra $U(g)$) to the group $*$-algebra $\mathbb{C}G$ generated (roughly speaking) by elements of G. This is for convenience since such exponentiated elements tend to be represented by bounded operators rather than unbounded ones. This is a standard construction in the theory of operator algebras (there is also a von Neumann algebra version) and X and G need only be locally compact rather than smooth manifolds. Any other quantization $\mathcal{A}$ is merely a $*$-representation of this cross product. The construction is roughly equivalent to Mackey's theory of systems of imprimitivity [7].

Many authors since Mackey have independently rediscovered this as a natural method of quantization of homogeneous spaces. See [9] for the version relevant to the present considerations of Hopf algebras. Specifically, we noted there that the notion of cross product is well-known to make sense when G is replaced by a non-cocommutative quantum group H such as (the operator-algebra version of) $U_q(g)$, and $C(X)$ by some non-commutative $*$-algebra. We have already given a canonical example of this type [9, example 4.7] where the cross product quantization happens again to be a Hopf algebra. We want to pursue this now for the quantum double.

Thus, we take for our 'position functions' some non-commutative $*$-algebra B, typically a q-deformation of some $C(X)$. For simplicity, we assume it is unital. We suppose that there is an action $\alpha = \triangleright$ of a Hopf $*$-algebra H on B such that

$$\begin{aligned} &h\triangleright(bc) = \sum (h_{(1)}\triangleright b)(h_{(2)}\triangleright c) \\ &h\triangleright 1 = \epsilon(h)1, \quad (h\triangleright b)^* = (Sh)^*\triangleright b^*, \quad h\in H,\ b\in B. \end{aligned} \tag{24}$$

The first conditions are the Hopf algebra analogue of the condition that H acts 'by automorphisms' of B and means that we can form a cross product algebra $B \rtimes H$ as explained in the preliminaries. It indeed has B, H as sub-algebras and cross relations in the equivalent form

$$\sum (1\otimes h_{(1)})(b\otimes 1)(1\otimes Sh_{(2)}) = (h\triangleright b\otimes 1), \quad h\in H,\ b\in B. \tag{25}$$

The last condition in (24) is a 'unitarity' condition arising from the requirement that $B \rtimes H$ is a $*$-algebra with B, H as $*$-subalgebras (just apply $*$ to (25) and require that we obtain (25) again for $((Sh)^*, b^*))$. This condition has already been noted in [21] where B is the quantum sphere and $H = U_q(\mathrm{su}_2)$. When it holds, $B \rtimes H$ becomes a $*$-algebra. We have also encountered both conditions in section 2. The quantization maps $\widehat{b} = b \otimes 1$ and $\widehat{h} = 1 \otimes h$ are then embeddings of B, H and in terms of them, and (25) becomes

$$\sum \widehat{h_{(1)}}\widehat{b}\widehat{Sh_{(2)}} = \widehat{h \triangleright b}, \qquad b \in B,\ h \in H \tag{26}$$

as a generalization of (23).

We have already seen in proposition 2.5 that the quantum adjoint action of H on itself obeys (24), so $H \rtimes H$ by this is a $*$-algebra. Likewise for several other canonical actions such as the left coregular action of H^* on H. The latter has quantization given by the cross product $H^* \rtimes H$ (generalizing the Weyl algebra on a group). Here $H^* \rtimes H \cong \mathrm{Lin}(H)$ but is not in general a Hopf algebra. Moreover, we have seen in proposition 2.6 that the coadjoint action of H on $\underline{A}$ also precisely obeys the conditions (24). This is the form that we need.

Corollary 3.1. *If H is a real-quasitriangular Hopf $*$-algebra in the sense $\mathcal{R}^{*\otimes *} = \mathcal{R}_{21}$ then $\underline{A} \rtimes H$ is a $*$-algebra cross product as above (a quantization). It is a Hopf algebra isomorphic to $D(H)$.*

Proof. This follows at once from proposition 2.6. Here $\underline{A}$ has its $*$-algebra structure $\underline{*}$ and H acts by the quantum coadjoint action. The $*$-structure on $\underline{A} \rtimes H$, which is the one relevant to the generalized Mackey quantization makes $\underline{A}$ and H $*$-subalgebras. The isomorphism $\theta : \underline{A} \rtimes H \to D(H)$ was obtained in [5,6] as

$$\theta(a \otimes h) = \sum a_{(1)} \langle \mathcal{R}^{(1)}, a_{(2)} \rangle \otimes \mathcal{R}^{(2)} h,$$
$$\theta^{-1}(a \otimes h) = \sum a_{(1)} \langle S\mathcal{R}^{(1)}, a_{(2)} \rangle \otimes \mathcal{R}^{(2)} h. \tag{27}$$

This isomorphism induces a $*$-algebra structure in $D(H)$ but note that it is quite different from that in example 2.1 where the subalgebras $H^{*\mathrm{op}}$ and H were $*$-subalgebras. In fact, the quantum mechanical $*$-structure from $\underline{A} \rtimes H$ does not naturally form a Hopf $*$-algebra – this is evident from the explicit form of the coproduct of $\underline{A} \rtimes H$ given in [6] as

$$\Delta(a \otimes h) = \sum a_{(1)} \otimes \mathcal{R}^{(2)} h_{(1)} \otimes \mathcal{R}^{(1)} \triangleright a_{(2)} \otimes h_{(2)}. \tag{28}$$

This is a cross coproduct by coaction $\beta(a) = \mathcal{R}_{21} \triangleright a$ where the second factor of $\mathcal{R}_{21}$ acts on a in $\underline{A}$ by the quantum coadjoint action.

We conclude from this that $D(H)$ in this case is the algebra of observables for the quantization of a particle on the non-commutative space $B = \underline{A}$ with momentum given by the quantum group H. This is a very general result. We concentrate now on the matrix case where $H = U_q(g)$ is in FRT form with generators $l^\pm$ and dual G_q of function algebra type. We have already recalled the braided groups BG_q from [20] as quotients of the braided-matrices $B(R)$ with matrix generator $\mathbf{u}$ (see the end of section 2).

Corollary 3.2. $\mathrm{BG}_q \rtimes U_q(g)$ *with q real is a $*$-algebra cross product describing a quantum particle with positions observables* BG_q *and momentum* $U_q(g)$*, and is a Hopf algebra isomorphic to* $D(U_q(g))$*. Explicitly, it has matrix generators u of* BG_q *and $l^\pm$ of $U_q(g)$ with cross relations and coproduct,*

$$Rl_2^+ u_1 = u_1 R l_2^+, \quad R_{21}^{-1} l_2^- u_1 = u_1 R_{21}^{-1} l_2^-,$$
$$\Delta l^\pm = l^\pm \otimes l^\pm, \quad \Delta u = \sum u\mathcal{R}^{(2)} \otimes \mathcal{R}^{(1)} \triangleright u,$$

where $\mathcal{R} = \sum \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}$ *is the universal R-matrix and* $\triangleright$ *is the quantum adjoint action. The $*$-structures are* ${u^i}_j^{\underline{*}} = {u^j}_i$, ${l^{\pm i}}_j^{\,*} = S{l^{\mp j}}_i$.

Proof. This is a special case of the last corollary. The quantum coadjoint action on BG_q coincides on the matrix generators with that on G_q (and with the quantum adjoint action on L in $U_q(g)$). It comes out as in [6] as

$$ {l^{+k}}_l \triangleright {u^i}_j = {R^{-1}}^i{}_a{}^k{}_b {u^a}_c R^c{}_j{}^b{}_l, \quad {l^{-i}}_j \triangleright {u^k}_l = R^i{}_a{}^k{}_b {u^b}_c {R^{-1}}^a{}_j{}^c{}_l \tag{29}$$

giving the relations shown, while the isomorphism (27) of the resulting cross product with $D(U_q(g))$ comes out as

$$\theta(1 \otimes l^\pm) = 1 \otimes l^\pm, \quad \theta({u^i}_j \otimes 1) = {t^i}_a \otimes S{l^{-a}}_j. \tag{30}$$

These steps are similar to those in [6, section 4]. Also, from corollary 3.1, it follows that we have a $*$-algebra cross product when q is real. We can also verify this directly using the reality property of the R-matrix in corollary 2.8,

$$\begin{aligned}({l^{+k}}_l \triangleright {u^i}_j)^{\underline{*}} &= \overline{{R^{-1}}^i{}_a{}^k{}_b}({u^a}_c)^{\underline{*}}\overline{R^c{}_j{}^b{}_l} = R^l{}_b{}^j{}_c {u^c}_a {R^{-1}}^b{}_k{}^a{}_i = {l^{-l}}_k \triangleright {u^j}_i \\ &= (S{l^{+k}}_l)^* \triangleright ({u^i}_j)^{\underline{*}}\end{aligned}$$

and similarly for l^-. Note that using the axioms for $\mathcal{R}$, one can also compute the coproduct further as

$$\Delta u = \left(\sum u\mathcal{R}^{(2)} \otimes \mathcal{R}^{(1)} u\right)\mathcal{R}_{21}^{-1}, \tag{31}$$

where the $\mathcal{R}$ live in the tensor square of $U_q(g)$.

This completes the general theory. To all the compact Lie group deformations we have associated quantum systems isomorphic as algebras to the quantum double. The position space is not an ordinary group or quantum group but a braided one. Moreover, its $*$-structure needed for our interpretation is transformed in the process from unitary to Hermitian in its general character. We conclude now with the full details in the simplest case, namely $g = \mathrm{su}_2$.

Firstly, the position observables is given by the algebra $BSU_q(2)$ with four generators (the matrix 'coordinates') and relations [20]

$$ba = q^2 ab, \qquad ca = q^{-2}ac, \qquad da = ad,$$

$$bc = cb + (1-q^{-2})a(d-a), \tag{32}$$

$$db = bd + (1-q^{-2})ab, \qquad cd = dc + (1-q^{-2})ca, \tag{33}$$

$$ad - q^2 cb = 1, \tag{34}$$

and equipped now with Hermitian $*$-structure as quoted in the introduction. If we write new self-adjoint generators

$$x_0 = qd + q^{-1}a, \quad x_1 = \frac{b+c}{2}, \quad x_2 = \frac{b-c}{2\mathrm{i}}, \quad x_3 = d-a \tag{35}$$

then x_0 (the time direction) is central and the left hand side of (34) is

$$\begin{aligned} ad - q^2 cb = {} & \frac{q^2}{(q^2+1)^2}x_0^2 - q^2 x_1^2 - q^2 x_2^2 \\ & - \frac{(q^4+1)q^2}{2(q^2+1)^2}x_3^2 + \left(\frac{q^2-1}{q^2+1}\right)^2 \frac{q}{2}x_0 x_3 \end{aligned} \tag{36}$$

so that setting this braided-determinant equal to 1 means that our position observables are a q-deformation of a hyperboloid or 3-sphere in Minkowski-space. This $*$-algebra can be denoted BS^3_q for this reason.

Secondly, our momentum group is $U_q(\mathrm{su}_2)$ with its usual $*$-structure. Its action on the position observables comes out from the above theory as [20]

$$X_+ \triangleright \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -q^{3/2}c & -q^{1/2}(d-a) \\ 0 & q^{-1/2}c \end{pmatrix} \to \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] \tag{37}$$

$$X_- \triangleright \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} q^{1/2}b & 0 \\ q^{-1/2}(d-a) & -q^{-3/2}b \end{pmatrix} \to \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right] \tag{38}$$

$$H \triangleright \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -2b \\ 2c & 0 \end{pmatrix} \to \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right], \tag{39}$$

where the limits are as $q \to 1$. These make clear that our action is a q-deformation of the one induced on the generators of the ring of co-ordinate functions by the adjoint action. At the group level this is the usual action induced on $C(S^3_{\mathrm{Lor}})$ by conjugation in SL_2. The orbits of this are spatial spheres.

The quantization of this system as a $*$-cross product is then isomorphic to the quantum double $D(U_q(\mathrm{su}_2))$. Thus the quantum double should be viewed as a q-deformed version of quantum motion on spheres. Of course, in the q-deformed setting there are neither actual points nor actual orbits in the usual sense. Moreover, one has to quantize the spheres together as a foliation of a hyperboloid in Minkowski space (rather than the more obvious setting of spheres in Euclidean space) for this interpretation to work in the q-deformed case.

This complete the details of the simplest model as discussed in (1) in the introduction. We note that the q that enters into the non-commutation relations (32), (33) should be thought of as due to braid statistics and not due to any quantization [20]. This is because the braided-group function algebra here is in a certain sense braided-commutative (cf. the super-commutativity of the coordinate ring for supergroups) and this is expressed by these relations. This point of view justifies our use of $\mathrm{BSU}_q(2) = \mathrm{BS}_q^3$ as the *classical* position space before quantization. The quantum group $U_q(\mathrm{su}_2)$ is likewise to be viewed as some kind of deformation, rather than quantization, of the classical angular momentum $U(\mathrm{su}_2)$. Although $\mathrm{BSU}_q(2)$ and $U_q(\mathrm{su}_2)$ are the same as $*$-algebras, their interpretation and coproducts are quite different. The quantum double is then the Mackey-type quantization of this q-deformed system.

Finally, we note that the models above can easily be extended using the same techniques. For example, we can relax (34) and thereby take for our position space a q-deformation of Minkowski space. In this case the algebra is isomorphic to the degenerate Sklyanin algebra as explained in [6] where also the action of $U_q(\mathrm{su}_2)$ is given. The momentum group can also be extended from $U_q(\mathrm{su}_2)$ to its quantum group complexification. For this one can take the quantum double $D(U_q(\mathrm{su}_2))$ regarded this time as a q-deformation of the Lorentz group [3]. This system will be explored elsewhere.

4. The dual of $D(H)$ as another quantum algebra of observables

The striking consequence of an algebra of observables being a Hopf algebra is that its dual (defined suitably in the infinite-dimensional case) is also a Hopf algebra. But is this the algebra of observables of some other quantum system, dual to the first? This was the case in the models in [9,14–16] and we will see that it is the case here also for the quantum double. The main reason is that $\underline{A} \rtimes H$ (isomorphic to the quantum double) has a crossed structure not only as an algebra but also as a coalgebra.

The axioms for the cross coproduct coalgebra structure were recalled in the preliminaries and are based on a left comodule coalgebra structure obeying (5) and dual to those for a left cross product algebra $B \rtimes H$ as in (3). Here it is an elementary fact that a left H-module algebra B with action $\triangleright$: $H \otimes B \to B$ dualises (at least in the finite-dimensional case) to a left A-comodule coalgebra C with coaction $\beta : C \to A \otimes C$ where C is the coalgebra dual to the algebra B and A a Hopf algebra dual to H. Likewise vice versa. Hence if, as for the double, B is both a left H-module algebra and a left H-comodule coalgebra then C is both a left A-comodule coalgebra and left A-module algebra. Moreover, if $B \rtimes H$ with the resulting crossed algebra and coalgebra structures turns out to be a Hopf algebra [34], clearly the dual algebra and coalgebra $C \rtimes A$ is then also a Hopf algebra. Applying this to the full Hopf algebra structure underlying corollary 3.1 we obtain

Proposition 4.1. *If A is an antireal dual-quasitriangular Hopf algebra dual to H then $(\underline{H})^{\mathrm{op/op}} \rtimes A$ is a $*$-algebra cross product, isomorphic to $D(H)^*$ as a Hopf algebra. Here $(\underline{H})^{\mathrm{op/op}}$ denotes the braided group $\underline{H}$ with the opposite product and coproduct.*

The detailed proof here requires us to compute the relevant dual as $(\underline{A})^* = (\underline{H})^{\mathrm{op/op}}$, as well as to analyse the condition needed for the resulting action of A (as obtained by dualization) to obey the unitarity condition as in (24) in order to have a $*$-algebra cross product. We do this now, but in some cleaner conventions that avoid the use of $(\ )^{\mathrm{op/op}}$ and corresponding complexities in the formulae: these are all avoided if in the course of dualization we make a further switch from left-modules and comodules to right-modules and comodules. Thus, for our explicit presentation of results in this section, we now switch over to these right-handed conventions.

Firstly, C is a right A-module algebra (where A is a Hopf algebra) if $\triangleleft$ is a right action and

$$(cd)\triangleleft a = \sum (c\triangleleft a_{(1)})(d\triangleleft a_{(2)}), \quad 1\triangleleft a = \epsilon(a)1. \tag{40}$$

In this case, there is a cross product algebra $C \rtimes A$ with

$$(c\otimes a)(d\otimes a') = \sum (c\triangleleft a'_{(2)})d\otimes a(a'_{(1)}). \tag{41}$$

Next, C is a right A-module coalgebra if it is a coalgebra and $\triangleleft : C\otimes A \to C$ is a coalgebra map. It is a right A-comodule algebra if it is an algebra and there is a right comodule β: $C \to C\otimes A$ which is an algebra map. Finally, it is a right A-comodule coalgebra if it is a coalgebra and β is a right comodule obeying

$$\begin{aligned}(\Delta_C\otimes \mathrm{id})\beta(c) &= \sum c_{(1)}{}^{\overline{(1)}}\otimes c_{(2)}{}^{\overline{(1)}}\otimes c_{(1)}{}^{\overline{(2)}}c_{(2)}{}^{\overline{(2)}},\\ (\epsilon_C\otimes \mathrm{id})\beta(c) &= 1\epsilon_C(c),\end{aligned} \tag{42}$$

where $\beta(c) = \sum c^{\overline{(1)}}\otimes c^{\overline{(2)}} \in C\otimes A$. In the last case there is a cross coproduct coalgebra $C \rtimes A$ with

$$\Delta(c\otimes a) = \sum c_{(1)}{}^{\overline{(1)}}\otimes a_{(1)}\otimes c_{(2)}\otimes a_{(2)}c_{(1)}{}^{\overline{(2)}}. \tag{43}$$

This summarises the right-handed version of the formulae in the preliminaries. Also, if A is a Hopf $*$-algebra and C an A-module algebra and $*$-algebra, for the cross product to be a $*$-algebra containing A, C as $*$-subalgebras, we require

$$(c\triangleleft a)^* = c^*\triangleleft(Sa)^*. \tag{44}$$

This is the right-handed analogue of the unitarity condition in (24).

Proposition 4.2. *(Cf. [34].) Let A be a Hopf algebra. If $(C, \underline{\Delta})$ is both a right A-module algebra and coalgebra, and a right A-comodule algebra and coalgebra and*

$$\underline{\Delta}(cd) = \sum c_{\underline{(1)}} \overline{d_{\underline{(1)}}}^{(1)} \otimes (c_{\underline{(2)}} \triangleleft \overline{d_{\underline{(1)}}}^{(2)}) d_{\underline{(2)}},$$
$$\sum (c \triangleleft a_{(2)})^{\overline{(1)}} \otimes a_{(1)} (c \triangleleft a_{(2)})^{\overline{(2)}} = \sum c^{\overline{(1)}} \triangleleft a_{(1)} \otimes c^{\overline{(2)}} a_{(2)}$$

along with $\underline{\Delta}1 = 1 \otimes 1$ etc., and a convolution-inverse of the identity $C \to C$, then $C \rtimes A$ by the cross product and coproduct, is a Hopf algebra.

Proof. This is nothing more than a right-handed version of an observation of Radford in [34]. Namely, as C is an A-module algebra, we have an associative cross product $C \rtimes A$, and since it is an A-comodule coalgebra, we have a coassociative cross coproduct. The remaining conditions ensure that these fit together to form a Hopf algebra. This is an elementary computation. In fact, there is a simple way to do all these dualization computations easily, by diagrammatic methods [29].

To understand why these right-handed conventions are appropriate in the dual, note first that the assertion that B is a left H-module algebra or coalgebra is equivalent to the assertion that its product or coproduct maps are intertwiners for the action of H, that is that B lives in the category of H-modules. In this setting B has a natural dual $B^\star$ which also lives in the same category. As a linear space it coincides with the usual dual B^*, but has the opposite coproduct or product to the usual one,

$$B^\star = B^{*\mathrm{op/op}}. \tag{45}$$

Moreover, $B^\star$ is again a left H-module under the action $\langle h \triangleright f, b\rangle = \langle f, (Sh) \triangleright b\rangle$ for all $b \in B$, $f \in B^\star$.

Lemma 4.3. *Let H be a finite-dimensional Hopf algebra with dual A. If B is a finite-dimensional left H-module algebra and H-comodule coalgebra, such that $B \rtimes H$ is a Hopf algebra, then $C = B^\star$ is in the situation of proposition 4.2 and*

$$C \rtimes A = (B \rtimes H)^\star \cong (B \rtimes H)^*.$$

Proof. One can see easily that if B is a left H-module (algebra, coalgebra) then $B^\star$ is a left H-module (coalgebra, algebra). Also, a left H-comodule (algebra, coalgebra) defines a right A-module (algebra, coalgebra) by usual dualization, where A is dual to H. Likewise, a left H-module (algebra, coalgebra) is equivalently a right A-comodule (algebra, coalgebra). Combining these elementary observations, we see that a left H-comodule (algebra, coalgebra) B has as its categorical dual $B^\star$ a right A-module (coalgebra, algebra), and a left H-module (algebra, coalgebra) has as its categorical dual a right A-comodule (coalgebra, algebra). Explicitly, the two are related by

$$\sum\langle b,c^{\overline{(1)}}\rangle\langle h,c^{\overline{(2)}}\rangle = \langle c,(Sh)\triangleright b\rangle,\quad \langle b,c\triangleleft(Sa)\rangle = \sum\langle a,b^{\overline{(1)}}\rangle\langle c,b^{\overline{(2)}}\rangle \tag{46}$$

for all $b\in B$, $c\in B^\star$, $h\in H$ and $a\in A = H^*$. From this and an elementary computation, one sees that if $B{\bowtie\!\!\!\triangleleft}H$ is a Hopf algebra (its module and comodules structures obey the original left-handed form [34] of the conditions in proposition 4.2) then the right handed $C\triangleright\!\!\!\triangleleft A$ is also a Hopf algebra by the proposition. Its natural pairing is with $(B\triangleright\!\!\!\triangleleft H)^*$ by the map

$$C\triangleright\!\!\!\triangleleft A\to(B\triangleright\!\!\!\triangleleft H)^*,\quad c\otimes a\mapsto\langle c\otimes S^{-1}a,\ \rangle;$$

$$\begin{aligned}
\langle(b\otimes h)(b'\otimes h'),c\otimes S^{-1}a\rangle &= \sum\langle b,c_{\underline{(2)}}\rangle\langle h_{(1)}\triangleright b',c_{\underline{(1)}}\rangle\langle h',S^{-1}a_{(1)}\rangle\langle h_{(2)},S^{-1}a_{(2)}\rangle\\
&= \sum\langle b',c_{\underline{(1)}}{}^{\overline{(1)}}\rangle\langle S^{-1}h_{(1)},c_{\underline{(1)}}{}^{\overline{(2)}}\rangle\langle b,c_{\underline{(2)}}\rangle\langle h',S^{-1}a_{(1)}\rangle\langle h_{(2)},S^{-1}a_{(2)}\rangle\\
&= \sum\langle b'\otimes h'\otimes b\otimes h,c_{\underline{(1)}}{}^{\overline{(1)}}\otimes S^{-1}a_{(1)}\otimes c_{\underline{(2)}}\otimes S^{-1}(a_{(2)}c_{\underline{(1)}}{}^{\overline{(2)}})\rangle\\
&= \langle b'\otimes h'\otimes b\otimes h,(\mathrm{id}\otimes S^{-1}\otimes\mathrm{id}\otimes S^{-1})\Delta_{C\triangleright\!\!\!\triangleleft A}c\otimes a\rangle
\end{aligned}$$

$$\begin{aligned}
\langle b\otimes h,(\mathrm{id}\otimes S^{-1})((c\otimes a)(c'\otimes a'))\rangle &= \sum\langle b\otimes h,(c\triangleleft a'_{(2)})c'\otimes(S^{-1}a'_{(1)})S^{-1}a\rangle\\
&= \sum\langle b_{\underline{(1)}},c'\rangle\langle b_{\underline{(2)}},c\triangleleft a'_{(2)}\rangle\langle h,(S^{-1}a'_{(1)})S^{-1}a\rangle\\
&= \sum\langle b_{\underline{(1)}},c'\rangle\langle S^{-1}a'_{(2)},b_{\underline{(2)}}{}^{\overline{(1)}}\rangle\langle b_{\underline{(2)}}{}^{\overline{(2)}},c\rangle\langle h_{(1)},S^{-1}a'_{(1)}\rangle\langle h_{(2)},S^{-1}a\rangle\\
&= \sum\langle c'\otimes S^{-1}a'\otimes c\otimes S^{-1}a,b_{\underline{(1)}}\otimes b_{\underline{(2)}}{}^{\overline{(1)}}h_{(1)}\otimes b_{\underline{(2)}}{}^{\overline{(2)}}\otimes h_{(2)}\rangle\\
&= \langle c'\otimes S^{-1}a'\otimes c\otimes S^{-1}a,\Delta_{B\triangleright\!\!\!\triangleleft H}b\otimes h\rangle
\end{aligned}$$

where $\Delta_B b = \sum b_{\underline{(1)}}\otimes b_{\underline{(2)}}$ and $\Delta_C c = \sum c_{\underline{(1)}}\otimes c_{\underline{(2)}}$ with $C = B^\star$ as in (45). Note that because $B\triangleright\!\!\!\triangleleft H$ is a Hopf algebra, its usual dual $(B\triangleright\!\!\!\triangleleft H)^*$ and $(B\triangleright\!\!\!\triangleleft H)^{*\mathrm{op/op}}$ are isomorphic via the antipode of $(B\triangleright\!\!\!\triangleleft H)^*$.

There is a similar result in the setting of dually-paired Hopf algebras. This takes care of the general constructions of which the quantum double and its dual (as we will see) are examples. We are now ready to study the situation when H is quasitriangular, with quasitriangular structure $\mathcal{R}$, or slightly more generally when A is dual quasitriangular with $\mathcal{R}\in(A\otimes A)^*$. The way that the cross product and coproduct structure of $D(H)$ was found in [5] was to show that if B is an H-module (algebra, coalgebra) then $\mathcal{R}$ can be used to also make B an H-comodule (algebra, coalgebra) and this led to $B\triangleright\!\!\!\triangleleft H$ as a Hopf algebra. The corresponding lemma in the dual language of $(A,\mathcal{R})$ is

Lemma 4.4. *(Cf. [5,24].) Let $(A,\mathcal{R})$ be dual quasitriangular. If C is a right A-comodule (algebra, coalgebra) then*

$$c\triangleleft a = \sum c^{\overline{(1)}}\mathcal{R}(c^{\overline{(2)}}\otimes a).$$

makes C also a right A-module (algebra, coalgebra) and the second condition stated in proposition 4.2 is satisfied. If C is a Hopf algebra living in the braided category of A-comodules then $C \rtimes\!\!\!\lhd A$ is a Hopf algebra by proposition 4.2.

Proof. The first part is nothing other than a dual version of an observation first made in [5, Prop. 3.1], and in any case follows at once from (10), (11). The second part is nothing other than a dual version of [24, Thm 4.1] (and corresponds to turning the diagram-proofs there upside down). It is also easy to see directly: For C a Hopf algebra living in the braided category of right A-comodules the condition that $\underline{\Delta}$: $C \to C\underline{\otimes}C$ is an algebra homomorphism to the braided tensor product [11],

$$\underline{\Delta}(cd) = \sum c_{\underline{(1)}} d_{\underline{(1)}}{}^{\overline{(1)}} \otimes c_{\underline{(2)}}{}^{\overline{(1)}} d_{\underline{(2)}} \mathcal{R}(c_{\underline{(2)}}{}^{\overline{(2)}} \otimes d_{\underline{(1)}}{}^{\overline{(2)}}) \tag{47}$$

reduces in the present context to the first condition of proposition 4.2. Hence, we have a biproduct $C \rtimes\!\!\!\lhd A$ forming a Hopf algebra. Note that the usual dualization of [24, Thm 4.1] is obtained trivially by dualizing the relevant structure maps (and adapting to the infinite-dimensional case), and does not require the right-handed theory above.

Lemma 4.5. *(Cf. [35].) Let H be a quasitriangular Hopf algebra with dual A and associated braided groups $\underline{H}, \underline{A}$. Then we have as Hopf algebras in the braided category of H-modules,*

$$\underline{H} \cong (\underline{A})^*.$$

Proof. We consider the map $\underline{H} \to (\underline{A})^*$ by $b \mapsto \langle Sb, \ \rangle$ and use (14), (15) to do all our computations in terms of the usual Hopf algebras H, A. Then

$$\begin{aligned}
\langle Sb, a\underline{\cdot}a'\rangle &= \sum \langle Sb_{(2)}, a_{(2)}\rangle \langle Sb_{(1)}, a'_{(2)}\rangle \mathcal{R}((Sa_{(1)})a_{(3)} \otimes Sa'_{(1)}) \\
&= \sum \langle (S\mathcal{R}^{(1)}{}_{(1)})(Sb_{(2)})\mathcal{R}^{(1)}{}_{(2)}, a\rangle \langle (S\mathcal{R}^{(2)})Sb_{(1)}, a'\rangle \\
&= \sum \langle S(b_{(1)}\mathcal{R}^{(2)}), a'\rangle \langle S(S^{-1}\mathcal{R}^{(1)}{}_{(2)} b_{(2)} \mathcal{R}^{(1)}{}_{(1)}), a\rangle \\
&= \sum \langle S(b_{(1)} S\mathcal{R}^{(2)}), a'\rangle \langle S(\mathcal{R}^{(1)}{}_{(1)} b_{(2)} S\mathcal{R}^{(1)}{}_{(2)}), a\rangle \\
&= \langle (S \otimes S)\underline{\Delta} b, a' \otimes a\rangle,
\end{aligned}$$

as required. The pairing $\langle Sb \otimes Sb', \underline{\Delta} a\rangle = \langle S(b'b), a\rangle$ is easier since the product of $\underline{H}$ and the coproduct of $\underline{A}$ coincide with the usual ones. Likewise for the pairing of the units and counits. We also check that the map $\langle S(\ ), \ \rangle$ is indeed a morphism in the category of H-modules,

$$\begin{aligned}
(\langle S(h \triangleright b), \ \rangle)(a) &= \langle S(h \triangleright b), a\rangle = \sum \langle h_{(1)} b S h_{(2)}, Sa\rangle \\
&= \sum \langle Sh_{(1)}, a_{(3)}\rangle \langle b, Sa_{(2)}\rangle \langle S^2 h_{(2)}, a_{(1)}\rangle
\end{aligned}$$

$$= \sum \langle Sb, a^{\overline{(1)}} \rangle \langle Sh, a^{\overline{(2)}} \rangle = \langle Sb, (Sh) \triangleright a \rangle$$
$$= (h \triangleright \langle Sb, \ \rangle)(a)$$

where the natural action on $f \in (\underline{A})^\star$ is $(h \triangleright f)(a) = f((Sh) \triangleright a)$ as explained above.

Putting together these various preliminary facts, we obtain

Proposition 4.6. *Let H be an antireal-quasitriangular Hopf algebra with dual A as in section 2, and $\underline{H}$ the braided group associated to H (here $\underline{H}$ coincides with H as a $*$-algebra). Then $\underline{H} \rtimes A$ is a (right handed) $*$-algebra cross product, and is a Hopf algebra isomorphic to $D(H)^*$. Here the action of A on $\underline{H}$ is*

$$b \triangleleft a = \sum \mathcal{R}^{(1)} \triangleright b \langle \mathcal{R}^{(2)}, a \rangle$$

where $\triangleright$ is the quantum adjoint action.

Proof. The general theory has been established above in the required form. We conclude from this that $\underline{H} \rtimes A$ is a right-handed cross product and coproduct, and forms a Hopf algebra dual to the left-handed $\underline{A} \rtimes H$ in corollary 3.1. It remains only to see how the action, coaction and isomorphism look explicitly and to see the condition needed for a $*$-algebra structure for the cross product. From (46) we have

$$\begin{aligned}
\langle S(b \triangleleft a), f \rangle &= \langle \langle Sb, \ \rangle \triangleleft a, f \rangle = \sum \langle S^{-1}a, \mathcal{R}^{(2)} \rangle \langle \mathcal{R}^{(1)} \triangleright f, Sb \rangle \\
&= \sum \langle S^{-1}a, \mathcal{R}^{(2)} \rangle \langle f_{(2)}, Sb \rangle \langle \mathcal{R}^{(1)}, (Sf_{(1)}) f_{(3)} \rangle \\
&= \sum \langle \mathcal{R}^{(2)}, a \rangle \langle (Sf)_{(2)}, b \rangle \langle \mathcal{R}^{(1)}, (Sf)_{(1)} S(Sf)_{(3)} \rangle \\
&= \sum \langle \mathcal{R}^{(2)}, a \rangle \langle S(\mathcal{R}^{(1)}{}_{(1)} b S \mathcal{R}^{(1)}{}_{(2)}), f \rangle
\end{aligned}$$

for all $f \in \underline{A}$. From this we conclude the form shown. In a similar way we can obtain the right coaction of A on $\underline{H}$ dual via (46) to the left coadjoint action of H on $\underline{A}$. It comes out as determined by

$$\sum b^{\overline{(1)}} \otimes \langle b^{\overline{(2)}}, h \rangle = h \triangleright b \qquad \forall h \in H, \ b \in \underline{H} \tag{48}$$

and is necessarily related to the right action $\triangleleft$ as in Lemma 4.4 with $C = \underline{H}$.

In a similar way one can trace through the definitions to deduce from the dual of (27) the isomorphism $\widehat{\theta} : \underline{H} \rtimes A \to D(H)^*$ as,

$$\widehat{\theta}(b \otimes a) = \langle \mathcal{R}^{(2)} e_{a(1)}, a \rangle (S\mathcal{R}^{(1)})(e_{a(2)} \triangleright b) \otimes f^a \tag{49}$$

where $\{e_a\}$ is a basis of H and $\{f^a\}$ is a dual one. This is obtained from the isomorphisms $\underline{H} \rtimes A \cong (\underline{A})^\star \rtimes A \cong D(H)^\star \cong D(H)^*$ by Lemma 4.5, Lemma 4.3 and corollary 3.1. The first isomorphism is via $\langle S(\ ), \ \rangle$, the second via $\langle \mathrm{id}\otimes S^{-1},\ \rangle$, the third via the adjoint of (27) and the last via the adjoint of the antipode of $D(H)$. Thus,

$$\begin{aligned}
\langle\widehat{\theta}(b\otimes a),a'\otimes h\rangle &= \langle b\otimes a,(S\otimes S^{-1})\circ\theta^{-1}\circ S_{D(H)}(a'\otimes h)\rangle\\
&= \langle b\otimes a,(S\otimes S^{-1})\circ\theta^{-1}((1\otimes Sh)(S^{-1}a'\otimes 1))\rangle\\
&= \sum\langle b\otimes a,(S\otimes S^{-1})\circ\theta^{-1}(S^{-1}a'_{(2)}\otimes Sh_{(2)})\rangle\\
&\quad\times\langle Sh_{(3)},a'_{(3)}\rangle\langle h_{(1)},a'_{(1)}\rangle\\
&= \sum\langle b\otimes a,a'_{(3)}\otimes h_{(2)}\mathcal{R}^{(2)}\rangle\langle h_{(1)},a'_{(1)}\rangle\\
&\quad\times\langle S\mathcal{R}^{(1)},a'_{(2)}\rangle\langle Sh_{(3)},a'_{(4)}\rangle\\
&= \sum\langle h_{(1)}(S\mathcal{R}^{(1)})bSh_{(3)},a'\rangle\langle a,h_{(2)}\mathcal{R}^{(2)}\rangle\\
&= \sum\langle (S\mathcal{R}^{(1)})h_{(2)}bSh_{(3)},a'\rangle\langle a,\mathcal{R}^{(2)}h_{(1)}\rangle\\
&= \sum\langle a,\mathcal{R}^{(2)}h_{(1)}\rangle\langle (S\mathcal{R}^{(1)})(h_{(2)}\triangleright b),a'\rangle
\end{aligned}$$

from which we deduce (49) at least in the finite-dimensional case.

Finally, we suppose that $\mathcal{R}$ is antireal. Then

$$\begin{aligned}
(b\triangleleft a)^* &= \sum\overline{\langle\mathcal{R}^{(2)},a\rangle}(\mathcal{R}^{(1)}\triangleright b)^* = \sum\langle(S\mathcal{R}^{(2)})^*,a^*\rangle(S\mathcal{R}^{(1)})^*\triangleright b^*\\
&= \sum\langle\mathcal{R}^{(2)*},a^*\rangle\mathcal{R}^{(1)*}\triangleright b^* = \sum\langle\mathcal{R}^{(2)},a^*\rangle(S\mathcal{R}^{(1)})\triangleright b^*\\
&= \sum\langle\mathcal{R}^{(2)},S^{-1}a^*\rangle\mathcal{R}^{(1)}\triangleright b^* = b^*\triangleleft S^{-1}(a^*) = b^*\triangleleft(Sa)^*,
\end{aligned}$$

as required. Hence, if A is antireal dual-quasitriangular, then $\underline{H}\rtimes A$ is a $*$-algebra cross product.

Note that the condition on $\mathcal{R}$ for a $*$-algebra cross product here is different from the one in corollary 3.1.

Example 4.7. $\mathrm{BU}_q(\mathrm{sl}(2,\mathbb{R}))\rtimes\mathrm{SL}_q(2,\mathbb{R})$ *with* $|q|=1$ *is a $*$-algebra cross product describing a quantum particle with momentum* $\mathrm{SL}_q(2,\mathbb{R})$ *and position observables* $\mathrm{BU}_q(\mathrm{sl}(2,\mathbb{R}))$, *and is a Hopf algebra isomorphic to the dual of* $D(U_q(\mathrm{sl}_2))$. *Explicitly, it has matrix generators L of* $\mathrm{BU}_q(\mathrm{sl}(2,\mathbb{R}))$ *and t of* $\mathrm{SL}_q(2,\mathbb{R})$ *with cross relations and coproduct*

$$L_1t_2=t_2R^{-1}L_1R,\qquad \Delta t=t\otimes t,\qquad \Delta L^i{}_j=L^a{}_b\otimes(St^i{}_a)t^b{}_cL^c{}_j.$$

The action and coaction here are $L_1\triangleleft t_2=R^{-1}L_1R$ *and* $\beta(L)=t^{-1}Lt$ *in the usual notations. The same result holds for any other* $U_q(g)$ *for which there is an antireal $*$-structure.*

Proof. This is a special case of the above proposition 4.6. We compute the action there as

$$L^i{}_j\triangleleft t^k{}_l=\mathcal{R}^{(1)}\triangleright L^i{}_j\langle\mathcal{R}^{(2)},t^k{}_l\rangle=l^{+k}{}_l\triangleright L^i{}_j \tag{50}$$

as already given in the proof of corollary 3.2. The coaction follows from (48) as

$$\beta(L^i{}_j) = L^a{}_b \otimes (St^i{}_a)t^b{}_j, \tag{51}$$

since this dualises to the left quantum adjoint action of H on $\underline{H}$. This is usually written compactly as conjugation by $\mathbf{t}$. From (41) and (43) we obtain the cross product and coproduct structures as stated. The isomorphism (49) with the dual of the double comes out as

$$\widehat{\theta}(L^i{}_j \otimes 1) = \sum e_a \triangleright L^i{}_j \otimes f^a, \quad \widehat{\theta}(1 \otimes t^i{}_j) = Sl^{+i}{}_k \otimes t^k{}_j. \tag{52}$$

Note finally that R is no longer of real type: For the sl_2 case at $|q| = 1$, it obeys

$$\overline{R^i{}_j{}^k{}_l} = R^{-1}{}^i{}_j{}^k{}_l. \tag{53}$$

Nevertheless, from example 2.3 and the general theory above, we know that we have a $*$-algebra cross product. One can also verify this example explicitly, as follows. Firstly, the $*$-structure on $U_q(\mathrm{sl}(2,\mathbb{R}))$ takes the form $l^{\pm i}{}_j{}^* = q^{i-j}l^{\pm i}{}_j$. From the pairing as a Hopf $*$-algebra with $A(R)$ one obtains likewise $t^i{}_j{}^* = q^{i-j}S^2t^i{}_j$ for the relevant $*$-structure for $\mathrm{SL}_q(2,\mathbb{R})$ (any even power of S will do here for a Hopf $*$-algebra, but this is the one that we need). This comes out as

$$\begin{pmatrix} t^1{}_1{}^* & t^1{}_2{}^* \\ t^2{}_1{}^* & t^2{}_2{}^* \end{pmatrix} = \begin{pmatrix} t^1{}_1 & qt^1{}_2 \\ q^{-1}t^2{}_1 & t^2{}_2 \end{pmatrix}.$$

Next, the action $L_1 \triangleleft t_2 = R^{-1}L_1R$ was already computed in a slightly more general context (for the degenerate Sklyanin algebra) in [6] as $(\ ) \triangleleft t^1{}_2 = 0$ and

$$\begin{pmatrix} q^{H/2} \\ q^{-H/2} \\ X_+ \\ X_- \end{pmatrix} \triangleleft t^1{}_1 = \begin{pmatrix} q^{H/2} \\ q^{-H/2} \\ qX_+ \\ q^{-1}X_- \end{pmatrix}, \qquad \begin{pmatrix} \ \end{pmatrix} \triangleleft t^1{}_1 = \begin{pmatrix} q^{H/2} \\ q^{-H/2} \\ q^{-1}X_+ \\ qX_- \end{pmatrix},$$

$$\begin{pmatrix} \ \end{pmatrix} \triangleleft t^2{}_1 = \begin{pmatrix} \lambda(1-q)X_+ \\ \lambda(1-q^{-1})X_+q^{-H} \\ \lambda(1-q^{-1})X_+^2q^{-H/2} \\ \lambda q^{-H/2}(X_+X_- - qX_-X_+) \end{pmatrix}$$

where $\lambda = q^{-1/2}(q - q^{-1})$. The non-trivial part of the verification of (44) is then

$$\left(\begin{pmatrix} q^{H/2} \\ q^{-H/2} \\ X_+ \\ X_- \end{pmatrix} \triangleleft t^2{}_1 \right)^* = \begin{pmatrix} \lambda(q-1)X_+ \\ \lambda(q^{-1}-1)X_+q^{-H} \\ \lambda(1-q^{-1})X_+^2q^{-H/2} \\ \lambda q^{-H/2}(X_+X_- - qX_-X_+) \end{pmatrix} = \begin{pmatrix} q^{H/2} \\ q^{-H/2} \\ -X_+ \\ -X_- \end{pmatrix} \triangleleft(-t^2{}_1)$$

$$= \begin{pmatrix} q^{H/2} \\ q^{-H/2} \\ X_+ \\ X_- \end{pmatrix}^* \triangleleft (St^2{}_1)^*.$$

This $*$-cross product is dual as a Hopf algebra to the quantum double model at the end of section 3. Nevertheless, we see that it has a similar interpretation although, this time, q is required to be of modulus 1 rather than real as before. To complete this picture it remains only to fill out the details of this interpretation.

Firstly, because $U_q(\mathrm{sl}_2)$ is factorizable, we have an isomorphism $\mathrm{BSL}_q(2) \cong \mathrm{BU}_q(\mathrm{sl}_2)$. The latter is equipped now with a $*$-structure that is that is inherited from $U_q(\mathrm{sl}(2,\mathbb{R}))$ and this means that $\mathrm{BSL}_q(2)$ also has a $*$-structure. From the explicit form of l^+Sl^- [33] in this case,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} q^H & q^{-1/2}(q-q^{-1})q^{H/2}X_- \\ q^{-1/2}(q-q^{-1})X_+q^{H/2} & q^{-H}+q^{-1}(q-q^{-1})^2X_+X_- \end{pmatrix}$$

one has

$$\begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} = \begin{pmatrix} a & q^2b \\ q^2c & q^2d+(1-q^2)a \end{pmatrix}. \tag{54}$$

We denote the braided group $\mathrm{BSL}_q(2)$ with this $*$-structure by $\mathrm{BSL}_q(2,\mathbb{R})$ in honour of the limit here as $q \to 1$. Its algebra relations are as in (32)–(34). This gives our interpretation of the position observables in example 4.7.

Secondly, we want to view the momentum quantum group $\mathrm{SL}_q(2,\mathbb{R})$ as a quantum enveloping algebra. This is not a new idea except that it is usually considered with the compact $*$-structure or without consideration of the $*$-structure at all. The algebra here is the usual $\mathrm{SL}_q(2)$ one [1,30]. If we define

$$q^\xi = t^1{}_1, \quad q^\eta = t^2{}_2, \quad \zeta = \frac{t^1{}_2}{q-q^{-1}}, \quad \chi = \frac{t^2{}_1}{q-q^{-1}}, \tag{55}$$

where we suppose that q is generic, then the algebra relations become

$$\begin{gathered}[\chi,\xi] = \chi = [\eta,\chi], \quad [\zeta,\xi] = \zeta = [\eta,\zeta], \quad [\chi,\zeta] = 0, \\ q^\xi q^\eta = 1 + (q-q^{-1})^2 q^{-1}\zeta\chi, \end{gathered} \tag{56}$$

while its usual matrix coproduct becomes

$$\Delta q^\xi = q^\xi \otimes q^\xi + (q-q^{-1})^2\zeta\otimes\chi, \quad \Delta q^\eta = q^\eta\otimes q^\eta + (q-q^{-1})^2\chi\otimes\zeta \tag{57}$$

$$\Delta\zeta = \zeta\otimes q^\eta + q^\xi\otimes\zeta, \quad \Delta\chi = \chi\otimes q^\xi + q^\eta\otimes\chi. \tag{58}$$

Finally, the $*$-structure for $\mathrm{SL}_q(2,\mathbb{R})$ used in example 4.7 becomes

$$\xi^* = -\xi, \quad \eta^* = -\eta, \quad \chi^* = -q^{-1}\chi, \quad \zeta^* = -q\zeta. \tag{59}$$

We assume that $q = e^t$ (where t in our case is imaginary) and deduce from the Campell–Baker–Hausdorf formula applied to these equations that

$$[\xi,\eta] = O(t), \quad \eta = -\xi + O(t).$$

The scaling of the generators is critical here and means that (with scaling as defined) we have in the limit the Lie algebra

$$[\chi,\xi] = \chi, \quad [\zeta,\xi] = \zeta, \quad [\chi,\zeta] = 0, \tag{60}$$

with its usual linear coproduct. This real Lie algebra is the solvable one appearing in the Iwasawa decomposition of sl_2 and can be called the Drinfeld dual su_2^* of su_2 in view of Drinfeld's general theory of Lie bialgebras [36]. Details of the computation from the Iwasawa decomposition can be found for example in [14, section 2]. In our case we see that the momentum quantum group in example 4.7 can be regarded as a q-deformation $U_q(\mathrm{su}_2^*)$ of its enveloping algebra, cf. the ideas introduced in [1, section 7].

Finally, the right action of this q-momentum group on the co-ordinate generators of $\mathrm{BSL}_q(2,\mathbb{R})$ is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \triangleleft \xi = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}, \quad (\ )\triangleleft\zeta = 0, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \triangleleft \chi = \begin{pmatrix} -qc & -(d-a) \\ 0 & q^{-1}c \end{pmatrix}. \tag{61}$$

In the limit $q \to 1$ we see that ξ, χ, ζ become anti-self-adjoint and have a $*$-representation on the algebra of functions on $\mathrm{SL}(2,\mathbb{R})$. This is the system whose quantization as a $*$-algebra semidirect product is the dual of the quantum double $D(U_q(\mathrm{sl}(2,\mathbb{R})))$ at least for generic q.

This completes the details of the dual model discussed in the Introduction in (2). Clearly, the physical meaning of these quantum systems is less familiar than those of the previous section. It is interesting however, that these dual systems have a mathematical interpretation as a $*$-algebra cross product provided $\mathcal{R}$ is antireal, whereas in section 3 it was required to be real. It seems that both cross products are not quantum $*$-algebras precisely at the same time, unless $\mathcal{R}$ is triangular (in which case the notions of real and antireal coincide).

5. Concluding remarks

For completeness we conclude by placing the above results in the context of two other interpretations of the quantum double (and of the semidirect products of the type above) that are developed elsewhere. These are semidirect products as a process of bosonization of braided objects [24] and semidirect products as quantum principal bundles [4].

The idea behind [24] is that a Hopf algebra in a braided category is analogous to the idea of a super or $\mathbb{Z}_2$-graded structure, with the role of the $\mathbb{Z}_2$ played by the quantum group H that generates the braided category (as explained in the preliminaries). Because of the grading there is necessarily a $\mathbb{Z}_2$-action and it is natural to 'bosonize' the super-Hopf algebra B into an ordinary Hopf algebra $B \rtimes\!\!\!\lhd \mathbb{Z}_2$. The idea is that the information previously in the grading or bose-fermi statistics is used expressed as non-commutativity in the algebra by adding an additional generator g with

$$bg = (-1)^{\deg(b)} gb, \qquad g^2 = 1.$$

This trick is well-known to physicists under the heading of the Jordan–Wigner transform [37] and also to mathematicians e.g. [38]. This is also the idea behind [24] where we showed that every braided-Hopf algebra in the category $\mathrm{Rep}(H)$ leads to an ordinary Hopf algebra $B \rtimes\!\!\!\!\!\lhd H$, its *bosonization.*

From this point of view, $D(H)$ is nothing other than the bosonization of the braided-group $\underline{A}$ of function algebra type associated to the quasitriangular Hopf algebra H. Here H is understood as generating the braiding or braid-statistics under which the braided-group is covariant. Thus two ideas, of grading and of momentum-covariance with associated Mackey quantization are unified when both are viewed in the general context of quantum groups. This gives insight into the nature of quantization as a process of braiding, and is explored further in [39].

One of the themes above has been the interaction of our constructions with Hopf algebra duality. Here we want to mention a powerful diagrammatic way of making such dualizations which is indispensable in a braided-group context [29] but useful even for Hopf algebras. The point is that the possibility of a dual Hopf algebra is based on the fact that the axioms of a Hopf algebra have an input-output symmetry in which the axiom system, when written as commuting diagrams, is invariant under reversal of the arrows. In the diagrammatic notation one goes further and writes all maps as nodes on strings flowing from the inputs down to the outputs. For example in [24] we gave the proofs of the bosonization construction in this way. Hence for the dual theorem one simply turns the diagram-proofs up-side-down. This turns left modules into right comodules etc. and recovers the general constructions of section 4 directly. From this point of view the content of Lemma 4.4 is precisely recovered as a dual version of the bosonization theorem: if C is a Hopf algebra living in the braided category of right comodules of a dual-quasitriangular Hopf algebra then it has a dual-bosonization $C \rtimes\!\!\!\!\!\lhd A$. This diagrammatic view of Hopf algebra duality is important also in the quantization interpretation where it suggests a kind of time-reversal and parity invariance of the system.

Finally, its is known from a general theorem of Radford [34] that simultaneous products and coproducts of the type above are in one-to-one correspondence with Hopf algebra projections. This means a Hopf algebra surjection $\pi\colon P \to A$ say where P, A are Hopf algebra and where the map π is split by a Hopf algebra inclusion $P \overset{i}{\hookleftarrow} A$ in such a way that $\pi \circ i = \mathrm{id}$. If these quantum groups are like functions on groups G, H respectively then π corresponds to an inclusion $H \subset G$. In this case one can view $G \to G/H$ as a principal H-bundle. In the same way, one can view $P \hookleftarrow B$ as a quantum principal bundle with structure quantum group A and base quantum space

$$B = P^A = \{b \in P |\ (\mathrm{id} \otimes \pi)\Delta b = b \otimes 1\}.$$

In the case where π is split the bundle is trivial with trivialization provided by i.

This means precisely that P factorises as $P = B \rtimes A$. We refer to [4] for further details.

Thus we see that semidirect products such as we have studied above can equally well be viewed as trivial quantum principal bundles. From this point of view, the quantum double and its dual define two quantum bundles, the first with structure quantum group $A = U_q(\mathrm{su}_2)$ which we need to view as a quantum function algebra of $\mathrm{SU}(2)^*$ and the second with structure quantum group $A = \mathrm{SL}_q(2,\mathbb{R})$. This time the perverse interpretation of an enveloping algebra as function algebra occurs with the model in section 3 rather than the dual model in section 4.

These mutually dual interpretations of the quantum double as bundles parallel then the mutually dual interpretations as quantization above. In summary, the quantum double, as well as the more general cross products and coproducts, allow us to extend the thesis of [9,14–17] that when both are sufficiently generalized, quantization and geometry are the same thing, from mutually dual points of view.

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